Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology

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Abstract: Lagrangian contact supersymmetries (depending on derivatives of arbitrary order) are treated in very general setting. The cohomology of the variational bicomplex on an arbitrary graded manifold and the iterated cohomology of a generic nilpotent contact supersymmetry are computed. In particular, the first variational formula and conservation laws for Lagrangian systems on graded manifolds using contact supersymmetries are obtained.

1 Introduction

At present, BRST transformations in the BV formalism [7, 24] provide the most interesting example of Lagrangian contact supersymmetries, depending on derivatives and preserving the contact ideal of graded exterior forms. Much that is already known regarding Lagrangian BRST theory (including the short variational complex, BRST cohomology [4, 5, 8], Noether's conservation laws [5, 16, 19]) has been formulated in terms of jet manifolds of vector bundles (see [5] for a survey) since the jet manifold formalism provides the algebraic description of Lagrangian and Hamiltonian systems of both even and odd variables. In spite of this formulation, most authors however assume the base manifold X of these bundles to be contractible because, e.g., the relative (local in the terminology of [4, 5]) cohomology are not trivial even when $X = \mathbb{R}^n$.

Stimulated by the BRST theory, we consider Lagrangian systems of odd variables and contact supersymmetries in very general setting. For this purpose, one usually calls into play fiber bundles over supermanifolds [12, 13, 17, 34]. We describe odd variables and their jets on an arbitrary smooth manifold X as generating elements of the structure ring of a graded manifold whose body is X [32, 38, 39]. This definition differs from that of jets of a graded fiber bundle [27], but reproduces the heuristic notion of jets of ghosts in the field-antifield BRST theory on $\mathbb{R}^n$ [5, 9].

Our goal is the following. Firstly, we construct the $\mathbb{Z}_2$-graded variational bicomplex on a graded manifold with an arbitrary body X, and obtain the cohomology of its short variational subcomplex and the complex of one-contact graded forms (Theorem 4.1). In particular, the first variational formula and conservation laws for Lagrangian systems on graded manifolds using contact supersymmetries are obtained (formulae (5.4) – (5.5)).

Secondly, the iterated cohomology of a generic nilpotent contact supersymmetry is computed (Theorems 6.2, 6.4 and 6.5). In the most interesting case of the form degree $n = \dim X$, it coincides with the above mentioned relative cohomology. Therefore, we extend the results of [5] and our recent work [21] to an arbitrary nilpotent contact supersymmetry.

As is well-known, generalized (depending on derivatives) symmetries of differential equations have been intensively investigated [3, 11, 29, 30, 35]. Generalized symmetries of Lagrangian systems on a local coordinate domain have been described in detail [11, 35]. The variational bicomplex constructed in the framework of the infinite order jet formalism enables one to provide the global analysis of Lagrangian systems on a fiber bundle and their symmetries [2, 22, 32, 42]. Sketched in Section 2 of our work, this analysis is extended to Lagrangian systems on graded manifolds (Section 3).

Recall that an r-order Lagrangian on a fiber bundle $Y \to X$ is defined as a horizontal density $L : J^rY \to \overset{n}{\wedge} T^*X$, $n = \dim X$, on the r-order jet manifold J^rY of sections of $Y \to X$. With the inverse system of finite order jet manifolds

$$X \overset{\pi}{\longleftarrow} Y \overset{\pi^1_0}{\longleftarrow} J^1Y \longleftarrow \cdots J^{r-1}Y \overset{\pi^r_{r-1}}{\longleftarrow} J^rY \longleftarrow \cdots, \tag{1.1}$$

we have the direct system

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^{1*}_0} \mathcal{O}^*_1 \longrightarrow \cdots \mathcal{O}^*_{r-1} \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \longrightarrow \cdots \tag{1.2}$$

of graded differential algebras (henceforth GDAs) of exterior forms on jet manifolds with respect to the pull-back monomorphisms $\pi^r_{r-1}{}^*$. Its direct limit is the GDA $\mathcal{O}^*_\infty$ consisting of all the exterior forms on finite order jet manifolds modulo the pull-back identification. The exterior differential on $\mathcal{O}^*_\infty$ is decomposed into the sum $d = d_H + d_V$ of the total and the vertical differentials. These differentials split $\mathcal{O}^*_\infty$ into a bicomplex. Introducing the projector ϱ (2.3) and the variational operator δ, one obtains the variational bicomplex (2.4) of $\mathcal{O}^*_\infty$. Its d_H- and δ-cohomology (Theorem 2.1) has been obtained in several steps [1, 2, 22, 35, 41, 42, 43].

In order to define the variational bicomplex on graded manifolds (Section 3), let us recall that, by virtue of Batchelor's theorem [6], any graded manifold $(\mathfrak{A}, X)$ with a body X is isomorphic to the one whose structure sheaf $\mathfrak{A}_Q$ is formed by germs of sections of the exterior product

$$\wedge Q^* = \mathbb{R} \underset{X}{\oplus} Q^* \underset{X}{\oplus} \overset{2}{\wedge} Q^* \underset{X}{\oplus} \cdots, \tag{1.3}$$

where Q^* is the dual of some real vector bundle $Q \to X$. In field models, a vector bundle Q is usually given from the beginning. Therefore, we consider graded manifolds $(X, \mathfrak{A}_Q)$ where Batchelor's isomorphism holds. We agree to call $(X, \mathfrak{A}_Q)$ the simple graded manifold constructed from Q. Accordingly, r-order jets of odd fields are defined as generating elements of the structure ring of the simple graded manifold $(X, \mathfrak{A}_{J^rQ})$ constructed from the jet bundle $J^rQ \to X$ of Q which is also a vector bundle [32, 39]. Let $\mathcal{C}^*_{J^rQ}$ be the bigraded differential algebra (henceforth BGDA) of $\mathbb{Z}_2$-graded (or, simply, graded) exterior forms on the graded manifold $(X, \mathfrak{A}_{J^rQ})$. A linear bundle morphism $\pi^r_{r-1} : J^rQ \to J^{r-1}Q$ yields the corresponding monomorphism of BGDAs $\mathcal{C}^*_{J^{r-1}Q} \to \mathcal{C}^*_{J^rQ}$ [6, 32]. Hence, there is the direct system of BGDAs

$$\mathcal{C}^*_Q \xrightarrow{\pi^{1*}_0} \mathcal{C}^*_{J^1Q} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{C}^*_{J^rQ} \longrightarrow \cdots, \tag{1.4}$$

whose direct limit $\mathcal{C}^*_\infty$ consists of graded exterior forms on graded manifolds $(X, \mathfrak{A}_{J^rQ})$, $r \in \mathbb{N}$, modulo the pull-back identification.

This definition of odd jets enables one to describe odd and even variables (e.g., fields, ghosts and antifields in BRST theory) on the same footing. Namely, let $Y \to X$ be an affine bundle and $\mathcal{P}^*_\infty \subset \mathcal{O}^*_\infty$ the $C^\infty(X)$-subalgebra of exterior forms whose coefficients are polynomial in the fiber coordinates on jet bundles $J^rY \to X$. This notion is intrinsic since any element of $\mathcal{O}^*_\infty$ is an exterior form on some finite order jet manifold and all jet bundles $J^rY \to X$ are affine. Let us consider the product $\mathcal{S}^*_\infty$ of graded algebras $\mathcal{C}^*_\infty$ and $\mathcal{P}^*_\infty$ over their common subalgebra $\mathcal{O}^*(X)$ of exterior forms on X. It is a BGDA which is split into the $\mathbb{Z}_2$-graded variational bicomplex, analogous to that of $\mathcal{O}^*_\infty$.

In Section 4, we obtain cohomology of some subcomplexes of the variational bicomplex $\mathcal{S}^*_\infty$ when X is an arbitrary manifold (Theorem 4.1). They are the short variational complex (4.1) of horizontal (local in the terminology of [5, 8]) graded exterior forms and the complex (4.2) of one-contact graded forms. For this purpose, one however must: (i) enlarge the BGDA $\mathcal{S}^*_\infty$ to the BGDA $\Gamma(\mathfrak{S}^*_\infty)$ of graded exterior forms of locally finite jet order, (ii) compute the cohomology of the corresponding complexes of $\Gamma(\mathfrak{S}^*_\infty)$, and (iii) prove that this cohomology of $\Gamma(\mathfrak{S}^*_\infty)$ coincides with that of $\mathcal{S}^*_\infty$. Following this procedure, we show that cohomology of the complex (4.1) equals the de Rham cohomology of X, while the complex (4.2) is globally exact.

Note that the exactness of the short variational complex (4.1) on $X = \mathbb{R}^n$ has been repeatedly proved [5, 8, 14]. One has also considered its subcomplex of graded exterior forms whose coefficients are constant on $\mathbb{R}^n$. Its d_H-cohomology is not trivial [5].

The exactness of the complex (4.2) enables us to generalize the first variational formula and Lagrangian conservation laws in the calculus of variations on fiber bundles to graded Lagrangians and contact supersymmetries of arbitrary order (Section 5).

Cohomology of the short variational complex (4.1) and its modification (6.2) is the main ingredient in a computation of the iterated cohomology of nilpotent contact supersymmetries.

By analogy with a contact symmetry (Proposition 2.3), an infinitesimal contact supertransformation or, simply, a contact supersymmetry υ is defined as a graded derivation of the $\mathbb{R}$-ring $\mathcal{S}^0_\infty$ such that the Lie derivative $\mathbf{L}_\upsilon$ preserves the contact ideal of the BGDA $\mathcal{S}^*_\infty$. The BRST transformation υ (5.7) in gauge theory on principal bundles exemplifies a first order contact supersymmetry such that the Lie derivative $\mathbf{L}_\upsilon$ of horizontal graded exterior forms is nilpotent. This fact motivates us to study nilpotent contact supersymmetries in a general setting.

The key point is that the Lie derivative $\mathbf{L}_\upsilon$ along a contact supersymmetry and the total differential d_H mutually commute. When $\mathbf{L}_\upsilon$ is nilpotent (Lemma 5.3), we suppose that the d_H-complex $\mathcal{S}^{0,*}_\infty$ of horizontal graded exterior forms is split into the bicomplex $\{S^{k,m}\}$ with respect to the nilpotent operator

$$\mathbf{s}_\upsilon\phi = (-1)^{|\phi|}\mathbf{L}_\upsilon\phi, \qquad \phi \in S^{0,*}_\infty, \tag{1.5}$$

and the total differential d_H. In the case of the above mentioned BRST transformation υ (5.7), $\mathbf{s}_\upsilon$ (1.5) is the BRST operator. One usually studies the relative cohomology $H^{*,*}(\mathbf{s}_\upsilon/d_H)$ of $\mathbf{s}_\upsilon$ with respect to the total differential d_H (see [15] for the BRST cohomology modulo the exterior differential d). This cohomology is not trivial even when $X = \mathbb{R}^n$, but it can be related to the total $(\mathbf{s}_\upsilon + d_H)$-cohomology only in the form degree $n = \dim X$. We consider the iterated cohomology $H^{*,*}(\mathbf{s}_\upsilon|d_H)$ of the bicomplex $\{S^{k,m}\}$ (Section 6). In the most interesting case of form degree $n = \dim X$, relative and iterated cohomology groups coincide. They

naturally characterize graded Lagrangians $L\in S^{*,n}$, for which υ is a variational symmetry, modulo Lie derivatives $\mathbf{L}_\upsilon\xi$, $\xi\in S^{0,*}_\infty$, and d_H-exact graded exterior forms.

Using the fact that d_H-cocycles are represented by exterior forms on X and that any exterior form on X is $\mathbf{s}_\upsilon$-closed, we obtain the iterated cohomology $H^{*,m<n}(\mathbf{s}_\upsilon|d_H)$ (Theorem 6.2) and state the relation between the iterated cohomology $H^{*,n}(\mathbf{s}_\upsilon|d_H)$ and the total $(\mathbf{s}_\upsilon+d_H)$-cohomology of the bicomplex $S^{*,*}$ (Theorems 6.4 and 6.5). Note that the relative cohomology has also been studied when the d_H-cohomology need not be trivial ([5], Section 9.6). Theorem 6.5 generalizes this and our result in [21] to an arbitrary nilpotent contact supersymmetry, when it may happen that an exterior form on X is $\mathbf{s}_\upsilon$-exact.

2 Lagrangian contact symmetries. Global analysis

Smooth manifolds throughout are real, finite-dimensional, Hausdorff, second-countable (hence, paracompact) and connected.

The projective limit $(J^\infty Y,\pi^\infty_r:J^\infty Y\to J^rY)$ of the inverse system (1.1) is a paracompact Fréchet manifold [42], called the infinite order jet manifold. A bundle coordinate atlas $\{(U_Y;x^\lambda,y^i)\}$ of $\pi:Y\to X$ yields the coordinate atlas

$$\{((\pi^\infty_0)^{-1}(U_Y);x^\lambda,y^i_\Lambda)\},\qquad {y'}^i_{\lambda+\Lambda}=\frac{\partial x^\mu}{\partial x'^\lambda}d_\mu y'^i_\Lambda,\qquad 0\leq|\Lambda|, \tag{2.1}$$

of $J^\infty Y$, where $\Lambda=(\lambda_k...\lambda_1)$ is a symmetric multi-index, $\lambda+\Lambda=(\lambda\lambda_k...\lambda_1)$, and

$$d_\lambda=\partial_\lambda+\sum_{|\Lambda|\geq 0}y^i_{\lambda+\Lambda}\partial^\Lambda_i,\qquad d_\Lambda=d_{\lambda_r}\circ\cdots\circ d_{\lambda_1},\quad \Lambda=(\lambda_r...\lambda_1), \tag{2.2}$$

are the total derivatives. Hereafter, we fix an atlas of Y and, consequently, that of $J^\infty Y$ containing a finite number of charts [26]. Restricted to the chart (2.1), the GDA $\mathcal{O}^*_\infty$ can be written in a coordinate form; horizontal forms $\{dx^\lambda\}$ and contact one-forms $\{\theta^i_\Lambda=dy^i_\Lambda-y^i_{\lambda+\Lambda}dx^\lambda\}$ make up a local basis for the $\mathcal{O}^0_\infty$-algebra $\mathcal{O}^*_\infty$.

There is the canonical decomposition $\mathcal{O}^*_\infty=\oplus\mathcal{O}^{k,m}_\infty$ of $\mathcal{O}^*_\infty$ into $\mathcal{O}^0_\infty$-modules $\mathcal{O}^{k,m}_\infty$ of k-contact and m-horizontal forms together with the corresponding projections $h_k:\mathcal{O}^*_\infty\to\mathcal{O}^{k,*}_\infty$ and $h^m:\mathcal{O}^*_\infty\to\mathcal{O}^{*,m}_\infty$. Accordingly, the exterior differential on $\mathcal{O}^*_\infty$ is split into the sum $d=d_H+d_V$ of the total and vertical differentials

$$\begin{array}{lll} d_H\circ h_k=h_k\circ d\circ h_k, & d_H\circ h_0=h_0\circ d, & d_H(\phi)=dx^\lambda\wedge d_\lambda(\phi),\\ d_V\circ h^m=h^m\circ d\circ h^m, & d_V(\phi)=\theta^i_\Lambda\wedge\partial^\Lambda_i\phi, & \phi\in\mathcal{O}^*_\infty. \end{array}$$

One also introduces the $\mathbb{R}$-module projector

$$\varrho=\sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n,\qquad \overline{\varrho}(\phi)=\sum_{|\Lambda|\geq 0}(-1)^{|\Lambda|}\theta^i\wedge[d_\Lambda(\partial^\Lambda_i\rfloor\phi)],\qquad \phi\in\mathcal{O}^{>0,n}_\infty, \tag{2.3}$$

of $\mathcal{O}^*_\infty$ such that $\varrho\circ d_H=0$ and the nilpotent variational operator $\delta=\varrho\circ d_V$ on $\mathcal{O}^{*,n}_\infty$. Put

$E_k = \varrho(\mathcal{O}^{k,n}_\infty)$. Then the GDA $\mathcal{O}^*_\infty$ is split into the variational bicomplex

$$
\begin{tikzcd}
 & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \\
 & 0 \arrow[r] & \mathcal{O}^{1,0}_\infty \arrow[u, "d_V"] \arrow[r, "d_H"] & \mathcal{O}^{1,1}_\infty \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots & \mathcal{O}^{1,m}_\infty \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots & \mathcal{O}^{1,n}_\infty \arrow[u, "d_V"] \arrow[r, "\varrho"] & E_1 \arrow[u, "-\delta"] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathcal{O}^{0}_\infty \arrow[u, "d_V"] \arrow[r, "d_H"] & \mathcal{O}^{0,1}_\infty \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots & \mathcal{O}^{0,m}_\infty \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots & \mathcal{O}^{0,n}_\infty \arrow[u, "d_V"] \arrow[r, phantom, "\equiv"] & \mathcal{O}^{0,n}_\infty \arrow[u, "-\delta"] & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathcal{O}^0(X) \arrow[u, "\pi^{\infty *}"] \arrow[r, "d"] & \mathcal{O}^1(X) \arrow[u, "\pi^{\infty *}"] \arrow[r, "d"] & \cdots & \mathcal{O}^m(X) \arrow[u, "\pi^{\infty *}"] \arrow[r, "d"] & \cdots & \mathcal{O}^n(X) \arrow[u, "\pi^{\infty *}"] \arrow[r, "d"] & 0 & \\
 & & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] & & 0 \arrow[u] & &
\end{tikzcd}
\tag{2.4}
$$

Its cohomology has been obtained in several steps (see the outline of proof of Theorem 2.1 in Appendix A).

Theorem 2.1. *(i) The second row from the bottom and the last column of this bicomplex make up the variational complex*

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \longrightarrow \cdots . \tag{2.5}$$

Its cohomology is isomorphic to the de Rham cohomology of the fiber bundle Y. (ii) The rows of contact forms of the bicomplex (2.4) are exact sequences.

One can think of the elements

$$L = \mathcal{L}\omega \in \mathcal{O}^{0,n}_\infty, \qquad \delta L = \sum_{|\Lambda|\geq 0} (-1)^{|\Lambda|} d_\Lambda(\partial^\Lambda_i \mathcal{L})\theta^i \wedge \omega \in E_1, \qquad \omega = dx^1 \wedge \cdots \wedge dx^n,$$

of the variational complex (2.5) as being a finite order Lagrangian and its Euler–Lagrange operator, respectively.

Corollary 2.2. *The exactness of the row of one-contact forms of the variational bicomplex (2.4) at the term $\mathcal{O}^{1,n}_\infty$ relative to the projector ϱ provides the $\mathbb{R}$-module decomposition*

$$\mathcal{O}^{1,n}_\infty = E_1 \oplus d_H(\mathcal{O}^{1,n-1}_\infty)$$

and, given a Lagrangian $L \in \mathcal{O}^{0,n}_\infty$, the corresponding decomposition

$$dL = \delta L - d_H \Xi. \tag{2.6}$$

The form Ξ in the decomposition (2.6) is not uniquely defined. It reads

$$\Xi = \sum_{s=0} F^{\lambda\nu_s\ldots\nu_1}_i \theta^i_{\nu_s\ldots\nu_1} \wedge \omega_\lambda, \quad F^{\nu_k\ldots\nu_1}_i = \partial^{\nu_k\ldots\nu_1}_i \mathcal{L} - d_\lambda F^{\lambda\nu_k\ldots\nu_1}_i + h^{\nu_k\ldots\nu_1}_i, \quad \omega_\lambda = \partial_\lambda \rfloor \omega,$$

where local functions $h \in \mathcal{O}^0_\infty$ obey the relations $h^\nu_i = 0$, $h^{(\nu_k\nu_{k-1})\ldots\nu_1}_i = 0$. It follows that $\Xi_L = \Xi + L$ is a Lepagean equivalent of a finite order Lagrangian L [25].

The decomposition (2.6) leads to the first variational formula (2.15) and the Lagrangian conservation law (2.16) as follows.

A derivation $\upsilon \in \mathfrak{d}\mathcal{O}^0_\infty$ of the $\mathbb{R}$-ring $\mathcal{O}^0_\infty$ such that the Lie derivative $\mathbf{L}_\upsilon$ preserves the contact ideal of the GDA $\mathcal{O}^*_\infty$ (i.e., the Lie derivative $\mathbf{L}_\upsilon$ of a contact form is a contact form) is called an infinitesimal contact transformation or, simply, a contact symmetry (by analogy with $\mathcal{C}$-transformations in [30] though υ need not come from a morphism of $J^\infty Y$). Proposition 2.3 below shows that, restricted to a coordinate chart (2.1) and a GDA $\mathcal{O}^*_r$ of finite jet order, any contact symmetry υ is the jet prolongation of a generalized vector field in [35].

Proposition 2.3. *(i) The derivation module $\mathfrak{d}\mathcal{O}^0_\infty$ is isomorphic to the $\mathcal{O}^0_\infty$-dual $(\mathcal{O}^1_\infty)^*$ of the module of one-forms $\mathcal{O}^1_\infty$. (ii) Relative to an atlas (2.1), a derivation $\upsilon \in \mathfrak{d}\mathcal{O}^0_\infty$ is given by the expression*

$$\upsilon = \upsilon^\lambda \partial_\lambda + \upsilon^i \partial_i + \sum_{|\Lambda|>0} \upsilon^i_\Lambda \partial^\Lambda_i, \tag{2.7}$$

where υ^λ, υ^i, υ^i_Λ are local smooth functions of finite jet order obeying the transformation law

$$\upsilon'^\lambda = \frac{\partial x'^\lambda}{\partial x^\mu}\upsilon^\mu, \qquad \upsilon'^i = \frac{\partial y'^i}{\partial y^j}\upsilon^j + \frac{\partial y'^i}{\partial x^\mu}\upsilon^\mu, \qquad \upsilon'^i_\Lambda = \sum_{|\Sigma|\leq|\Lambda|} \frac{\partial y'^i_\Lambda}{\partial y^j_\Sigma}\upsilon^j_\Sigma + \frac{\partial y'^i_\Lambda}{\partial x^\mu}\upsilon^\mu. \tag{2.8}$$

(iii) A derivation υ (2.7) is a contact symmetry iff

$$\upsilon^i_\Lambda = d_\Lambda(\upsilon^i - y^i_\mu \upsilon^\mu) + y^i_{\mu+\Lambda}\upsilon^\mu, \qquad 0 < |\Lambda|. \tag{2.9}$$

Proof. (i) At first, let us show that $\mathcal{O}^*_\infty$ is generated by elements df, $f \in \mathcal{O}^0_\infty$. It suffices to justify that any element of $\mathcal{O}^1_\infty$ is a finite $\mathcal{O}^0_\infty$-linear combination of elements df, $f \in \mathcal{O}^0_\infty$. Indeed, every $\phi \in \mathcal{O}^1_\infty$ is an exterior form on some finite order jet manifold J^rY. By virtue of the Serre–Swan theorem extended to non-compact manifolds [36, 40], the $C^\infty(J^rY)$-module $\mathcal{O}^*_r$ of one-forms on J^rY is a projective module of finite rank, i.e., ϕ is represented by a finite $C^\infty(J^rY)$-linear combination of elements df, $f \in C^\infty(J^rY) \subset \mathcal{O}^0_\infty$. Any element $\Phi \in (\mathcal{O}^1_\infty)^*$ yields a derivation $\upsilon_\Phi(f) = \Phi(df)$ of the $\mathbb{R}$-ring $\mathcal{O}^0_\infty$. Since the module $\mathcal{O}^1_\infty$ is generated by elements df, $f \in \mathcal{O}^0_\infty$, different elements of $(\mathcal{O}^1_\infty)^*$ provide different derivations of $\mathcal{O}^0_\infty$, i.e., there is a monomorphism $(\mathcal{O}^1_\infty)^* \to \mathfrak{d}\mathcal{O}^0_\infty$. By the same formula, any derivation $\upsilon \in \mathfrak{d}\mathcal{O}^0_\infty$ sends $df \mapsto \upsilon(f)$ and, since $\mathcal{O}^0_\infty$ is generated by elements df, it defines a morphism $\Phi_\upsilon : \mathcal{O}^1_\infty \to \mathcal{O}^0_\infty$. Moreover, different derivations υ provide different morphisms Φ_υ. Thus, we have a monomorphism and, consequently, an isomorphism $\mathfrak{d}\mathcal{O}^0_\infty \to (\mathcal{O}^1_\infty)^*$.

(ii) Restricted to a coordinate chart (2.1), $\mathcal{O}^1_\infty$ is a free $\mathcal{O}^0_\infty$-module generated by the exterior forms dx^λ, θ^i_Λ. Then $\mathfrak{d}\mathcal{O}^0_\infty = (\mathcal{O}^1_\infty)^*$ restricted to this chart consists of elements (2.7), where ∂_λ, ∂^Λ_i are the duals of dx^λ, θ^i_Λ. The transformation rule (2.8) results from the transition functions (2.1). The interior product $\upsilon\rfloor\phi$ and the Lie derivative $\mathbf{L}_\upsilon\phi$, $\phi \in \mathcal{O}^*_\infty$, obey the standard formulae. Restricted to a coordinate chart, the Lie derivative $\mathbf{L}_\upsilon$ sends each finite jet order GDA $\mathcal{O}^*_r$ to another finite jet order GDA $\mathcal{O}^*_s$. Since the atlas (2.1) is finite, $\mathbf{L}_\upsilon\phi$ preserves $\mathcal{O}^*_\infty$.

(iii) The expression (2.9) results from a direct computation similar to that of the first part of Bäcklund's theorem [29]. One can then justify that local functions (2.9) satisfy the transformation law (2.8). □

Any contact symmetry admits the horizontal splitting

$$\upsilon = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + (\vartheta^i \partial_i + \sum_{|\Lambda|>0} d_\Lambda \vartheta^i \partial_i^\Lambda), \qquad \vartheta^i = \upsilon^i - y^i_\mu \upsilon^\mu, \tag{2.10}$$

relative to the canonical connection $\nabla = dx^\lambda \otimes d_\lambda$ on the $C^\infty(X)$-ring $\mathcal{O}^0_\infty$ [32].

Lemma 2.4. *Any vertical contact symmetry $\upsilon = \upsilon_V$ obeys the relations*

$$\upsilon\rfloor d_H\phi = -d_H(\upsilon\rfloor\phi), \tag{2.11}$$
$$\mathbf{L}_\upsilon(d_H\phi) = d_H(\mathbf{L}_\upsilon\phi), \qquad \phi\in\mathcal{O}^*_\infty. \tag{2.12}$$

Proof. It is easily justified that, if ϕ and ϕ' satisfy the relation (2.11), then $\phi\wedge\phi'$ does well. Then it suffices to prove the relation (2.11) when ϕ is a function and $\phi = \theta^i_\Lambda$. The result follows from the equalities

$$\upsilon\rfloor\theta^i_\Lambda = \upsilon^i_\Lambda, \qquad d_H(\upsilon^i_\Lambda) = \upsilon^i_{\lambda+\Lambda}dx^\lambda, \qquad d_H\theta^i_\lambda = dx^\lambda\wedge\theta^i_{\lambda+\Lambda}, \tag{2.13}$$
$$d_\lambda\circ\upsilon^i_\Lambda\partial_i^\Lambda = \upsilon^i_\Lambda\partial_i^\Lambda\circ d_\lambda. \tag{2.14}$$

The relation (2.12) is a corollary of the equality (2.11). □

Proposition 2.5. *Given a Lagrangian $L = \mathcal{L}\omega \in \mathcal{O}^{0,n}_\infty$, its Lie derivative $\mathbf{L}_\upsilon L$ along a contact symmetry υ (2.10) fulfils the first variational formula*

$$\mathbf{L}_\upsilon L = \upsilon_V\rfloor\delta L + d_H(h_0(\upsilon\rfloor\Xi_L)) + \mathcal{L}d_V(\upsilon_H\rfloor\omega), \tag{2.15}$$

where Ξ_L is a Lepagean equivalent, e.g., a Poincaré–Cartan form of L.

Proof. The formula (2.15) comes from the splitting (2.6) and the relation (2.11) as follows:

$$\begin{aligned}\mathbf{L}_\upsilon L &= \upsilon\rfloor dL + d(\upsilon\rfloor L) = [\upsilon_V\rfloor dL - d_V\mathcal{L}\wedge\upsilon_H\rfloor\omega] + [d_H(\upsilon_H\rfloor L) + d_V(\mathcal{L}\upsilon_H\rfloor\omega)] = \\ &\upsilon_V\rfloor dL + d_H(\upsilon_H\rfloor L) + \mathcal{L}d_V(\upsilon_H\rfloor\omega) = \upsilon_V\rfloor\delta L - \upsilon_V\rfloor d_H\Xi + d_H(\upsilon_H\rfloor L) + \mathcal{L}d_V(\upsilon_H\rfloor\omega) = \\ &\upsilon_V\rfloor\delta L + d_H(\upsilon_V\rfloor\Xi + \upsilon_H\rfloor L) + \mathcal{L}d_V(\upsilon_H\rfloor\omega),\end{aligned}$$

where $\upsilon_V\rfloor\Xi = h_0(\upsilon\rfloor\Xi)$ since Ξ is a one-contact form, $\upsilon_H\rfloor L = h_0(\upsilon\rfloor L)$, and $\Xi_L = \Xi + L$. □

Let υ be a variational symmetry of L (in the terminology of [35]), i.e., $\mathbf{L}_\upsilon L = d_H\sigma$, $\sigma\in\mathcal{O}^{0,n-1}_\infty$. By virtue of the expression (2.15), this condition implies that υ is projected onto X. Then the first variational formula (2.15) restricted to $\mathrm{Ker}\,\delta L$ leads to the weak conservation law

$$0 \approx d_H(h_0(\upsilon\rfloor\Xi_L) - \sigma). \tag{2.16}$$

3 $\mathbb{Z}_2$-Graded variational bicomplex

Let $(X, \mathfrak{A}_Q)$ be the simple graded manifold constructed from a vector bundle $Q \to X$ of fiber dimension m. Its structure ring $\mathcal{A}_Q$ of sections of $\mathfrak{A}_Q$ consists of sections of the exterior bundle (1.3) called graded functions. Given bundle coordinates (x^λ, q^a) on Q with transition functions $q'^a = \rho^a_b q^b$, let $\{c^a\}$ be the corresponding fiber bases for $Q^* \to X$, together with transition functions $c'^a = \rho^a_b c^b$. Then (x^λ, c^a) is called the local basis for the graded manifold $(X, \mathfrak{A}_Q)$ [6, 32]. With respect to this basis, graded functions read

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1\ldots a_k} c^{a_1} \cdots c^{a_k},$$

where $f_{a_1\cdots a_k}$ are local smooth real functions on X.

Given a graded manifold $(X, \mathfrak{A}_Q)$, by the sheaf $\mathfrak{d}\mathfrak{A}_Q$ of graded derivations of $\mathfrak{A}_Q$ is meant a subsheaf of endomorphisms of the structure sheaf $\mathfrak{A}_Q$ such that any section u of $\mathfrak{d}\mathfrak{A}_Q$ over an open subset $U \subset X$ is a $\mathbb{Z}_2$-graded derivation of the $\mathbb{Z}_2$-graded ring $\mathcal{A}_Q(U)$ of graded functions on U, i.e.,

$$u(ff') = u(f)f' + (-1)^{[u][f]} f u(f'), \qquad f, f' \in \mathcal{A}_Q(U),$$

where [.] denotes the Grassmann parity. One can show that sections of $\mathfrak{d}\mathfrak{A}_Q$ over U exhaust all $\mathbb{Z}_2$-graded derivations of the ring $\mathcal{A}_Q(U)$ [6]. Let $\mathfrak{d}\mathcal{A}_Q$ be the Lie superalgebra of $\mathbb{Z}_2$-graded derivations of the $\mathbb{R}$-ring $\mathcal{A}_Q$. Its elements are called $\mathbb{Z}_2$-graded (or, simply, graded) vector fields on $(X, \mathfrak{A}_Q)$. Due to the canonical splitting $VQ = Q \times Q$, the vertical tangent bundle $VQ \to Q$ of $Q \to X$ can be provided with the fiber bases $\{\partial_a\}$ which is the dual of $\{c^a\}$. Then a graded vector field takes the local form $u = u^\lambda \partial_\lambda + u^a \partial_a$, where u^λ, u^a are local graded functions. It acts on $\mathcal{A}_Q$ by the rule

$$u(f_{a\ldots b} c^a \cdots c^b) = u^\lambda \partial_\lambda (f_{a\ldots b}) c^a \cdots c^b + u^d f_{a\ldots b} \partial_d \rfloor (c^a \cdots c^b). \tag{3.1}$$

This rule implies the corresponding transformation law

$$u'^\lambda = u^\lambda, \qquad u'^a = \rho^a_j u^j + u^\lambda \partial_\lambda (\rho^a_j) c^j.$$

Then one can show [32, 38] that graded vector fields on a simple graded manifold can be represented by sections of the vector bundle $\mathcal{V}_Q \to X$ which is locally isomorphic to the vector bundle $\wedge Q^* \otimes_X (Q \oplus_X TX)$, and is equipped with the bundle coordinates $(\dot{x}^\lambda_{a_1\ldots a_k}, v^i_{b_1\ldots b_k})$, $k = 0, \ldots, m$, together with the transition functions

$$\dot{x}'^\lambda_{i_1\ldots i_k} = \rho^{-1}{}^{a_1}_{i_1} \cdots \rho^{-1}{}^{a_k}_{i_k} \dot{x}^\lambda_{a_1\ldots a_k},$$
$$v'^i_{j_1\ldots j_k} = \rho^{-1}{}^{b_1}_{j_1} \cdots \rho^{-1}{}^{b_k}_{j_k} \left[\rho^i_j v^j_{b_1\ldots b_k} + \frac{k!}{(k-1)!} \dot{x}^\lambda_{b_1\ldots b_{k-1}} \partial_\lambda \rho^i_{b_k} \right].$$

Using this fact, we can introduce graded exterior forms on the simple graded manifold $(X, \mathfrak{A}_Q)$ as sections of the exterior bundle $\wedge \mathcal{V}^*_Q$, where $\mathcal{V}^*_Q \to X$ is the pointwise $\wedge Q^*$-dual of $\mathcal{V}_Q$. Relative to the dual bases $\{dx^\lambda\}$ for T^*X and $\{dc^b\}$ for Q^*, graded one-forms read

$$\phi = \phi_\lambda dx^\lambda + \phi_a dc^a, \qquad \phi'_a = \rho^{-1}{}^b_a \phi_b, \qquad \phi'_\lambda = \phi_\lambda + \rho^{-1}{}^b_a \partial_\lambda (\rho^a_j) \phi_b c^j.$$

The duality morphism is given by the interior product

$$u\rfloor\phi = u^\lambda\phi_\lambda + (-1)^{[\phi_a]}u^a\phi_a.$$

Graded exterior forms constitute the BGDA $\mathcal{C}^*_Q$ with respect to the bigraded exterior product $\wedge$ and the exterior differential d. The standard formulae of a BGDA hold.

Since the jet bundle $J^rQ \to X$ of a vector bundle $Q \to X$ is a vector bundle, let us consider the simple graded manifold $(X, \mathfrak{A}_{J^rQ})$ constructed from $J^rQ \to X$. Its local basis is $\{x^\lambda, c^a_\Lambda\}$, $0 \leq |\Lambda| \leq r$, together with the transition functions

$$c'^a_{\lambda+\Lambda} = d_\lambda(\rho^a_j c^j_\Lambda), \qquad d_\lambda = \partial_\lambda + \sum_{|\Lambda|<r} c^a_{\lambda+\Lambda}\partial^\Lambda_a, \tag{3.2}$$

where ∂^Λ_a are the duals of c^a_Λ. Let $\mathcal{C}^*_{J^rQ}$ be the BGDA of graded exterior forms on the graded manifold $(X, \mathfrak{A}_{J^rQ})$. The direct limit $\mathcal{C}^*_\infty$ of the direct system (1.4) inherits the BGDA operations intertwined by the monomorphisms $\pi^r_{r-1}{}^*$. It is locally a free $C^\infty(X)$-algebra countably generated by the elements $(1, c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}dx^\lambda)$, $0 \leq |\Lambda|$.

It should be emphasized that, in contrast with the GDA $\mathcal{O}^*_\infty$, the BGDA $\mathcal{C}^*_\infty$ consists of sections of sheaves over X. In order to regard these algebras on the same footing, let $Y \to X$ hereafter be an affine bundle. Then one can show that the GDA $\mathcal{O}^*_\infty$ is an algebra of sections of some sheaf over X (see Appendix B). Let us consider the above mentioned polynomial subalgebra $\mathcal{P}^*_\infty$ of $\mathcal{O}^*_\infty$ and the product $\mathcal{C}^*_\infty \wedge \mathcal{P}^*_\infty$ of graded algebras $\mathcal{C}^*_\infty$ and $\mathcal{P}^*_\infty$ over their common graded subalgebra $\mathcal{O}^*(X)$ of exterior forms on X. It consists of the elements

$$\sum_i \psi_i \otimes \phi_i, \qquad \sum_i \phi_i \otimes \psi_i, \qquad \psi \in \mathcal{C}^*_\infty, \qquad \phi \in \mathcal{P}^*_\infty,$$

of the tensor products $\mathcal{C}^*_\infty \otimes \mathcal{P}^*_\infty$ and $\mathcal{P}^*_\infty \otimes \mathcal{C}^*_\infty$ of the $C^\infty(X)$-modules $\mathcal{C}^*_\infty$ and $\mathcal{P}^*_\infty$ which are subject to the commutation relations

$$\begin{aligned} &\psi \otimes \phi = (-1)^{|\psi||\phi|}\phi \otimes \psi, \qquad \psi \in \mathcal{C}^*_\infty, \qquad \phi \in \mathcal{P}^*_\infty, \\ &(\psi \wedge \sigma) \otimes \phi = \psi \otimes (\sigma \wedge \phi), \qquad \sigma \in \mathcal{O}^*(X), \end{aligned} \tag{3.3}$$

and the multiplication

$$(\psi \otimes \phi) \wedge (\psi' \otimes \phi') := (-1)^{|\psi'||\phi|}(\psi \wedge \psi') \otimes (\phi \wedge \phi'). \tag{3.4}$$

Elements $\psi \otimes \phi$ are endowed with the total form degree $|\psi| + |\phi|$ and the total Grassmann parity $[\psi]$. Then the multiplication (3.4) obeys the relation

$$\varphi \wedge \varphi' = (-1)^{|\varphi||\varphi'|+[\varphi][\varphi']}\varphi' \wedge \varphi, \qquad \varphi, \varphi' \in \mathcal{S}^*_\infty,$$

and makes $\mathcal{C}^*_\infty \wedge \mathcal{P}^*_\infty$ into a bigraded $C^\infty(X)$-algebra $\mathcal{S}^*_\infty$, where the asterisk means the total form degree. Due to the algebra monomorphisms

$$\mathcal{C}^*_\infty \ni \psi \to \psi \otimes 1 = 1 \otimes \psi \in \mathcal{S}^*_\infty, \qquad \mathcal{P}^*_\infty \ni \phi \to \phi \otimes 1 = 1 \otimes \phi \in \mathcal{S}^*_\infty,$$

one can think of $\mathcal{S}^*_\infty$ as being an algebra generated by elements of $\mathcal{C}^*_\infty$ and $\mathcal{P}^*_\infty$. For instance, elements of the ring S^0_∞ are polynomials of c^a_Λ and y^i_Λ with coefficients in $C^\infty(X)$.

Let us provide $\mathcal{S}^*_\infty$ with the exterior differential

$$d(\psi\otimes\phi) := (d_{\mathcal{C}}\psi)\otimes\phi + (-1)^{|\psi|}\psi\otimes(d_{\mathcal{P}}\phi), \qquad \psi\in\mathcal{C}^*_\infty, \qquad \phi\in\mathcal{P}^*_\infty, \tag{3.5}$$

where $d_{\mathcal{C}}$ and $d_{\mathcal{P}}$ are exterior differentials on the differential algebras $\mathcal{C}^*_\infty$ and $\mathcal{P}^*_\infty$, respectively. We obtain at once from the relation (3.3) that

$$d(\phi\otimes\psi) = (d_{\mathcal{P}}\phi)\otimes\psi + (-1)^{|\phi|}\phi\otimes(d_{\mathcal{C}}\psi), \qquad \psi\in\mathcal{C}^*_\infty, \qquad \phi\in\mathcal{P}^*_\infty.$$

The exterior differential d (3.5) is nilpotent. It obeys the equalities

$$d(\varphi\wedge\varphi') = d\varphi\wedge\varphi' + (-1)^{|\varphi|}\varphi\wedge d\varphi', \qquad \varphi,\varphi'\in\mathcal{S}^*_\infty,$$

and makes $\mathcal{S}^*_\infty$ into a BGDA, which is locally generated by the elements

$$(1, c^a_\Lambda, y^i_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}dx^\lambda, \theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda), \qquad |\Lambda|\geq 0.$$

Hereafter, let the collective symbols s^A_Λ and θ^A_Λ stand both for even and odd generating elements c^a_Λ, y^i_Λ, θ^a_Λ, θ^i_Λ of the $C^\infty(X)$-algebra $\mathcal{S}^*_\infty$ which, thus, is locally generated by $(1, s^A_\Lambda, dx^\lambda, \theta^A_\Lambda)$, $|\Lambda|\geq 0$. We agree to call elements of $\mathcal{S}^*_\infty$ the graded exterior forms on X.

Similarly to $\mathcal{O}^*_\infty$, the BGDA $\mathcal{S}^*_\infty$ is decomposed into $\mathcal{S}^0_\infty$-modules $\mathcal{S}^{k,r}_\infty$ of k-contact and r-horizontal graded forms together with the corresponding projections h_k and h^r. Accordingly, the exterior differential d (3.5) on $\mathcal{S}^*_\infty$ is split into the sum $d = d_H + d_V$ of the total and vertical differentials

$$d_H(\phi) = dx^\lambda\wedge d_\lambda(\phi), \qquad d_V(\phi) = \theta^A_\Lambda\wedge\partial^\Lambda_A\phi, \qquad \phi\in\mathcal{S}^*_\infty.$$

The projection endomorphism ϱ of $\mathcal{S}^*_\infty$ is given by the expression

$$\varrho = \sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n, \qquad \overline{\varrho}(\phi) = \sum_{|\Lambda|\geq 0}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda(\partial^\Lambda_A\rfloor\phi)], \qquad \phi\in\mathcal{S}^{>0,n}_\infty,$$

similar to (2.3). The graded variational operator $\delta = \varrho\circ d$ is introduced. Then the BGDA $\mathcal{S}^*_\infty$ is split into the $\mathbb{Z}_2$-graded variational bicomplex

$$(\mathcal{O}^*(X), \mathcal{S}^{*,*}_\infty, E_k = \varrho(\mathcal{S}^{k,n}_\infty); d, d_H, d_V, \varrho, \delta), \tag{3.6}$$

analogous to the variational bicomplex (2.4).

4 Cohomology of $\mathbb{Z}_2$-graded complexes

We aim to study the cohomology of the short variational complex

$$0\longrightarrow\mathbb{R}\longrightarrow\mathcal{S}^0_\infty\xrightarrow{d_H}\mathcal{S}^{0,1}_\infty\cdots\xrightarrow{d_H}\mathcal{S}^{0,n}_\infty\xrightarrow{\delta}E_1 \tag{4.1}$$

and the complex of one-contact graded forms

$$0\to\mathcal{S}^{1,0}_\infty\xrightarrow{d_H}\mathcal{S}^{1,1}_\infty\cdots\xrightarrow{d_H}\mathcal{S}^{1,n}_\infty\xrightarrow{\varrho}E_1\to 0 \tag{4.2}$$

of the BGDA $\mathcal{S}^*_\infty$. One can think of the elements

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_\infty, \qquad \delta(L) = \sum_{|\Lambda|\geq 0} (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A L) \in E_1$$

of the complexes (4.1) – (4.2) as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

Theorem 4.1. *The cohomology of the complex (4.1) equals the de Rham cohomology $H^*(X)$ of X. The complex (4.2) is exact.*

The proof of Theorem 4.1 follows the scheme of the proof of Theorem 2.1, but all sheaves are sheaves over X. The proof falls into the three steps.

(i) We start by showing that the complexes (4.1) – (4.2) are locally exact.

Lemma 4.2. *The complex (4.1) on $X = \mathbb{R}^n$ is exact.*

Referring to [5], Theorems 4.1 – 4.2, for the proof, we summarize a few formulae quoted in the sequel. Any horizontal graded form $\phi \in \mathcal{S}^{0,*}_\infty$ admits the decomposition

$$\phi = \phi_0 + \widetilde\phi, \qquad \widetilde\phi = \int_0^1 \frac{d\lambda}{\lambda} \sum_{|\Lambda|\geq 0} s^A_\Lambda \partial^\Lambda_A \phi, \tag{4.3}$$

where ϕ_0 is an exterior form on $\mathbb{R}^n$. Let $\phi \in \mathcal{S}^{0,m<n}_\infty$ be d_H-closed. Then its component ϕ_0 (4.3) is an exact exterior form on $\mathbb{R}^n$ and $\widetilde\phi = d_H\xi$, where ξ is given by the following expressions. Let us introduce the operator

$$D^{+\nu}\widetilde\phi = \int_0^1 \frac{d\lambda}{\lambda} \sum_{k\geq 0} k\delta^\nu_{(\mu_1}\delta^{\alpha_1}_{\mu_2}\cdots\delta^{\alpha_{k-1}}_{\mu_k)}\lambda s^A_{(\alpha_1\ldots\alpha_{k-1})}\partial^{\mu_1\ldots\mu_k}_A\widetilde\phi(x^\mu, \lambda s^A_\Lambda, dx^\mu). \tag{4.4}$$

The relation $[D^{+\nu}, d_\mu]\widetilde\phi = \delta^\nu_\mu\widetilde\phi$ holds, and leads to the desired expression

$$\xi = \sum_{k=0} \frac{(n-m-1)!}{(n-m+k)!} D^{+\nu}P_k\partial_\nu\rfloor\widetilde\phi, \qquad P_0 = 1, \quad P_k = d_{\nu_1}\cdots d_{\nu_k}D^{+\nu_1}\cdots D^{+\nu_k}. \tag{4.5}$$

Now let $\phi \in \mathcal{S}^{0,m<n}_\infty$ be a graded density such that $\delta\phi = 0$. Then its component ϕ_0 (4.3) is an exact form on $\mathbb{R}^n$ and $\widetilde\phi = d_H\xi$, where ξ is given by the expression

$$\xi = \sum_{|\Lambda|\geq 0}\sum_{\Sigma+\Xi=\Lambda} (-1)^{|\Sigma|} s^A_\Xi d_\Sigma \partial^{\mu+\Lambda}_A \widetilde\phi\omega_\mu. \tag{4.6}$$

Remark 4.1. Since elements of $\mathcal{S}^*_\infty$ are polynomials in s^A_Λ, the sum in the expression (4.5) is finite. However, the expression (4.5) contains a d_H-exact summand which prevents its extension to $\mathcal{O}^*_\infty$. In this respect, we also quote the homotopy operator (5.107) in [35] which leads to the expression

$$\xi = \int_0^1 I(\phi)(x^\mu, \lambda s^A_\Lambda, dx^\mu)\frac{d\lambda}{\lambda}, \tag{4.7}$$

$$I(\phi) = \sum_{|\Lambda|\geq 0}\sum_\mu \frac{\Lambda_\mu + 1}{n-m+|\Lambda|+1} d_\Lambda[\sum_{|\Xi|\geq 0}(-1)^\Xi \frac{(\mu+\Lambda+\Xi)!}{(\mu+\Lambda)!\Xi!} s^A d_\Xi \partial^{\mu+\Lambda+\Xi}_A(\partial_\mu\rfloor\phi)], \tag{4.8}$$

where $\Lambda! = \Lambda_{\mu_1}! \cdots \Lambda_{\mu_n}!$ and Λ_μ denotes the number of occurrences of the index μ in Λ. The graded forms (4.6) and (4.7) differ in a d_H-exact graded form.

Lemma 4.3. *The complex (4.2) on $X = \mathbb{R}^n$ is exact.*

Proof. The fact that a d_H-closed graded $(1,m)$-form $\phi \in \mathcal{S}^{1,m<n}_\infty$ is d_H-exact is derived from Lemma 4.2 as follows. We write

$$\phi = \sum \phi^\Lambda_A \wedge \theta^A_\Lambda, \tag{4.9}$$

where $\phi^\Lambda_A \in \mathcal{S}^{0,m}_\infty$ are horizontal graded m-forms. Let us introduce additional variables $\overline{s}^A_\Lambda$ of the same Grassmann parity as s^A_Λ. Then one can associate to each graded $(1,m)$-form ϕ (4.9) a unique horizontal graded m-form

$$\overline{\phi} = \sum \phi^\Lambda_A \overline{s}^A_\Lambda, \tag{4.10}$$

whose coefficients are linear in the variables $\overline{s}^A_\Lambda$, and *vice versa*. Let us consider the modified total differential

$$\overline{d}_H = d_H + dx^\lambda \wedge \sum_{|\Lambda|>0} \overline{s}^A_{\lambda+\Lambda} \overline{\partial}^\Lambda_A,$$

acting on graded forms (4.10), where $\overline{\partial}^\Lambda_A$ is the dual of $d\overline{s}^A_\Lambda$. Comparing the equality $\overline{d}_H \overline{s}^A_\Lambda = dx^\lambda s^A_{\lambda+\Lambda}$ and the last equality (2.13), one can easily justify that $\overline{d_H\phi} = \overline{d}_H \overline{\phi}$. Let a graded $(1,m)$-form ϕ (4.9) be d_H-closed. Then the associated horizontal graded m-form $\overline{\phi}$ (4.10) is $\overline{d}_H$-closed and, by virtue of Lemma 4.2, it is $\overline{d}_H$-exact, i.e., $\overline{\phi} = \overline{d}_H \overline{\xi}$, where $\overline{\xi}$ is a horizontal graded $(m-1)$-form given by the expression (4.5) depending on additional variables $\overline{s}^A_\Lambda$. A glance at this expression shows that, since $\overline{\phi}$ is linear in the variables $\overline{s}^A_\Lambda$, so is $\overline{\xi} = \sum \xi^\Lambda_A \overline{s}^A_\Lambda$. It follows that $\phi = d_H \xi$ where $\xi = \sum \xi^\Lambda_A \wedge \theta^A_\Lambda$. It remains to prove the exactness of the complex (4.2) at the last term E_1. If

$$\varrho(\sigma) = \sum_{|\Lambda|\geq 0} (-1)^{|\Lambda|} \theta^A \wedge [d_\Lambda(\partial^\Lambda_A \rfloor \sigma)] = \sum_{|\Lambda|\geq 0} (-1)^{|\Lambda|} \theta^A \wedge [d_\Lambda \sigma^\Lambda_A] \omega = 0, \qquad \sigma \in \mathcal{S}^{1,n}_\infty,$$

a direct computation gives

$$\sigma = d_H \xi, \qquad \xi = -\sum_{|\Lambda|\geq 0} \sum_{\Sigma+\Xi=\Lambda} (-1)^{|\Sigma|} \theta^A_\Xi \wedge d_\Sigma \sigma^{\mu+\Lambda}_A \omega_\mu. \tag{4.11}$$

□

Remark 4.2. The proof of Lemma 4.3 fails to be extended to complexes of higher contact forms because the products $\theta^A_\Lambda \wedge \theta^B_\Sigma$ and $s^A_\Lambda s^B_\Sigma$ obey different commutation rules.

(ii) Let us associate to each open subset $U \subset X$ the BGDA $\mathcal{S}^*_U$ of elements of the $C^\infty(X)$-algebra $\mathcal{S}^*_\infty$ whose coefficients are restricted to U. These algebras make up a presheaf over X. Let $\mathfrak{S}^*_\infty$ be the sheaf of germs of this presheaf and $\Gamma(\mathfrak{S}^*_\infty)$ its structure module of sections. One can show that $\mathfrak{S}^*_\infty$ inherits the variational bicomplex operations, and $\Gamma(\mathfrak{S}^*_\infty)$ does so (see Appendix C). For short, we say that $\Gamma(\mathfrak{S}^*_\infty)$ consists of polynomials in s^a_Λ, ds^a_Λ of locally

bounded jet order $|\Lambda|$. There is the monomorphism $\mathcal{S}^*_\infty \to \Gamma(\mathfrak{S}^*_\infty)$. Let us consider the complexes of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{S}^0_\infty \xrightarrow{d_H} \mathfrak{S}^{0,1}_\infty \cdots \xrightarrow{d_H} \mathfrak{S}^{0,n}_\infty \xrightarrow{\delta} \mathfrak{E}_1, \qquad \mathfrak{E}_1 = \varrho(\mathfrak{S}^{1,n}_\infty), \tag{4.12}$$

$$0 \to \mathfrak{S}^{1,0}_\infty \xrightarrow{d_H} \mathfrak{S}^{1,1}_\infty \cdots \xrightarrow{d_H} \mathfrak{S}^{1,n}_\infty \xrightarrow{\varrho} \mathfrak{E}_1 \to 0 \tag{4.13}$$

over X and the complexes of their structure modules

$$0 \longrightarrow \mathbb{R} \longrightarrow \Gamma(\mathfrak{S}^0_\infty) \xrightarrow{d_H} \Gamma(\mathfrak{S}^{0,1}_\infty) \cdots \xrightarrow{d_H} \Gamma(\mathfrak{S}^{0,n}_\infty) \xrightarrow{\delta} \Gamma(\mathfrak{E}_1), \tag{4.14}$$

$$0 \to \Gamma(\mathfrak{S}^{1,0}_\infty) \xrightarrow{d_H} \Gamma(\mathfrak{S}^{1,1}_\infty) \cdots \xrightarrow{d_H} \Gamma(\mathfrak{S}^{1,n}_\infty) \xrightarrow{\varrho} \Gamma(\mathfrak{E}_1) \to 0. \tag{4.15}$$

By virtue of Lemmas 4.2 – 4.3 and Theorem 8.3, the complexes of sheaves (4.12) – (4.13) are exact. The terms $\mathfrak{S}^{*,*}_\infty$ of the complexes (4.12) – (4.13) are sheaves of $C^\infty(X)$-modules. Therefore, they are fine and, consequently, acyclic. By virtue of Theorem 8.4 (see Appendix D), the cohomology of the complex (4.14) equals the cohomology of X with coefficients in the constant sheaf $\mathbb{R}$, i.e., the de Rham cohomology $H^*(X)$ of X, whereas the complex (4.15) is globally exact.

(iii) It remains to prove the following.

Proposition 4.4. *Cohomology of the complexes (4.1) – (4.2) equals that of the complexes (4.14) – (4.15).*

Let the common symbol D stand for the operators d_H, δ and ϱ in the complexes (4.14) – (4.15), and let Γ^*_∞ denote the terms of these complexes. Since cohomology groups of these complexes are either trivial or equal to the de Rham cohomology of X, one can say that any D-closed element $\phi \in \Gamma^*_\infty$ takes the form

$$\phi = \psi + D\xi, \qquad \xi \in \Gamma^*_\infty, \tag{4.16}$$

where ψ is a closed exterior form on X which is not necessarily exact. Since all D-closed elements of Γ^*_∞ of finite jet order are also of form (4.16), it suffices to show that, if an element $\phi \in \mathcal{S}^*_\infty$ is D-exact in the module Γ^*_∞ (i.e., $\phi = D\xi$, $\xi \in \Gamma^*_\infty$), then it is also in $\mathcal{S}^*_\infty$ (i.e., $\phi = D\varphi$, $\varphi \in \mathcal{S}^*_\infty$).

Let X be a (contractible) domain, and let an element $\phi \in \mathcal{S}^*_\infty$ be D-exact in Γ^*_∞. Then, being D-closed, it is D-exact in $\mathcal{S}^*_\infty$ in accordance with Lemmas 4.2 and 4.3. Moreover, a glance at the expressions (4.5), (4.6) and (4.8) shows that the maximal jet order $[\varphi]$ of φ is bounded by an integer $N([\phi])$ which depends only on the maximal jet order $[\phi]$ of ϕ. It follows that, if $\phi = D\varphi$ is an arbitrary D-exact form of the jet order less than k, then the jet order of φ does not exceed $N(k)$. We agree to call this fact the finite exactness of an operator D.

Let X be an arbitrary manifold and U a domain of X. By virtue of Lemmas 4.2 and 4.3, the restriction of the operator D to $\Gamma^*_\infty|_U$ (or, roughly speaking, the operator D on U) has the finite exactness property. Let us state the following.

Lemma 4.5. *Given a family $\{U_\alpha\}$ of disjoint open subsets of X, let us suppose that the finite exactness of the operator D takes place on each subset U_α separately. Then D on the union $\bigcup\limits_\alpha U_\alpha$ also has the finite exactness property.*

Proof. Let $\phi \in \mathcal{S}^*_\infty$ be a D-exact graded form on X. The finite exactness on $\cup U_\alpha$ holds since $\phi = D\varphi_\alpha$ on every U_α and all $[\varphi_\alpha] < N([\phi])$. □

Lemma 4.6. *Suppose that the finite exactness of an operator D takes place on open subsets U, V of X and their non-empty overlap $U \cap V$. Then it is also true on $U \cup V$.*

Proof. Let $\phi = D\varphi \in \mathcal{S}^*_\infty$ be a D-exact graded form on X. By assumption, it can be brought into the form $D\varphi_U$ on U and $D\varphi_V$ on V, where φ_U and φ_V are graded forms of bounded jet order. Due to the decomposition (4.16), one can choose the forms φ_U, φ_V such that $\varphi - \varphi_U$ on U and $\varphi - \varphi_V$ on V are D-exact. Let us consider the difference $\varphi_U - \varphi_V$ on $U \cap V$. It is a D-exact graded form of bounded jet order which, by assumption, can be written as $\varphi_U - \varphi_V = D\sigma$ where σ is also of bounded jet order. Lemma 4.7 below shows that $\sigma = \sigma_U + \sigma_V$ where σ_U and σ_V are graded forms of bounded jet order on U and V, respectively. Then, putting

$$\varphi'_U = \varphi_U - D\sigma_U, \qquad \varphi'_V = \varphi_V + D\sigma_V,$$

we have the graded form ϕ, equal to $D\varphi'_U$ on U and $D\varphi'_V$ on V, respectively. Since the difference $\varphi'_U - \varphi'_V$ on $U \cap V$ vanishes, we obtain $\phi = D\varphi'$ on $U \cup V$ where

$$\varphi' \stackrel{\text{def}}{=} \begin{cases} \varphi'|_U = \varphi'_U \\ \varphi'|_V = \varphi'_V \end{cases}$$

is of bounded jet order. □

Lemma 4.7. *Let U and V be open subsets of X and σ a graded form of bounded jet order on $U \cap V$. Then σ splits into the sum $\sigma_U + \sigma_V$ of graded exterior forms σ_U on U and σ_V on V of bounded jet order.*

Proof. By taking a smooth partition of unity on $U \cup V$ subordinate to its cover $\{U, V\}$ and passing to the function with support in V, we get a smooth real function f on $U \cup V$ which is 0 on a neighborhood U_{U-V} of $U - V$ and 1 on a neighborhood U_{V-U} of $V - U$ in $U \cup V$. The graded form $f\sigma$ vanishes on $U_{U-V} \cap (U \cap V)$ and, therefore, can be extended by 0 to U. Let us denote it σ_U. Accordingly, the graded form $(1 - f)\sigma$ has an extension σ_V by 0 to V. Then $\sigma = \sigma_U + \sigma_V$ is a desired decomposition because σ_U and σ_V are of finite jet order which does not exceed that of σ. □

Lemma 9.5 in [10], Chapter V, states that, if some property holds on a domain and obeys the conditions of Lemma 4.5 and 4.6, it holds on any open subset of $\mathbb{R}^n$. Hence, the operator D has the jet exactness property on any open subset of $\mathbb{R}^n$ and, consequently, on any chart of the fiber bundle $Q \times_X Y \to X$. Since the latter admits a finite bundle atlas with the transition functions (2.1) and (3.2) preserving the jet order, the finite exactness of D takes place on the whole manifold X in accordance with Lemma 4.6. This proves Proposition 4.4 and, consequently, Theorem 4.1.

Remark 4.3. Let us consider the complex

$$0 \to \mathbb{R} \longrightarrow \mathcal{S}^0_\infty \xrightarrow{d} \mathcal{S}^1_\infty \cdots \xrightarrow{d} \mathcal{S}^k_\infty \longrightarrow \cdots, \tag{4.17}$$

which we agree to call the de Rham complex because $(\mathcal{S}^*_\infty, d)$ is the differential calculus over the $\mathbb{R}$-ring $\mathcal{S}^0_\infty$. If $X=\mathbb{R}^n$, it is exact ([14], Theorem 3.1). Similarly to the proof of Theorem 4.1, one can show that the cohomology of the de Rham complex (4.17) equals the de Rham cohomology of X.

Corollary 4.8. *Every d_H-closed graded form $\phi\in\mathcal{S}^{0,m<n}_\infty$ falls into the sum*

$$\phi=\psi+d_H\xi, \qquad \xi\in\mathcal{S}^{0,m-1}_\infty, \tag{4.18}$$

where ψ is a closed m-form on X. Every δ-closed graded Lagrangian $L\in\mathcal{S}^{0,n}_\infty$ is the sum

$$\phi=\psi+d_H\xi, \qquad \xi\in\mathcal{S}^{0,n-1}_\infty, \tag{4.19}$$

where ψ is a non-exact n-form on X.

The global exactness of the complex (4.2) at the term $\mathcal{S}^{1,n}_\infty$ results in the following.

Proposition 4.9. *Given a graded Lagrangian $L=\mathcal{L}\omega$, there is the decomposition*

$$dL=\delta L-d_H\Xi, \qquad \Xi\in\mathcal{S}^{1,n-1}_\infty, \tag{4.20}$$

$$\Xi=\sum_{s=0}\theta^A_{\nu_s\ldots\nu_1}\wedge F^{\lambda\nu_s\ldots\nu_1}_A\omega_\lambda, \qquad F^{\nu_k\ldots\nu_1}_A=\partial^{\nu_k\ldots\nu_1}_A\mathcal{L}-d_\lambda F^{\lambda\nu_k\ldots\nu_1}_A+h^{\nu_k\ldots\nu_1}_A, \tag{4.21}$$

where local graded functions h obey the relations $h^\nu_a=0$, $h^{(\nu_k\nu_{k-1})\ldots\nu_1}_a=0$.

Proof. The decomposition (4.20) is a straightforward consequence of the exactness of the complex (4.2) at the term $\mathcal{S}^{1,n}_\infty$ and the fact that ϱ is a projector. The coordinate expression (4.21) results from a direct computation

$$\begin{aligned}
-d_H\Xi = {} & -d_H[\theta^AF^\lambda_A+\theta^A_\nu F^{\lambda\nu}_A+\cdots+\theta^A_{\nu_s\ldots\nu_1}F^{\lambda\nu_s\ldots\nu_1}_A+\theta^A_{\nu_{s+1}\nu_s\ldots\nu_1}\wedge F^{\lambda\nu_{s+1}\nu_s\ldots\nu_1}_A+\cdots]\wedge\omega_\lambda= \\
& [\theta^Ad_\lambda F^\lambda_A+\theta^A_\nu(F^\nu_A+d_\lambda F^{\lambda\nu}_A)+\cdots+\theta^A_{\nu_{s+1}\nu_s\ldots\nu_1}(F^{\nu_{s+1}\nu_s\ldots\nu_1}_A+d_\lambda F^{\lambda\nu_{s+1}\nu_s\ldots\nu_1}_A)+\cdots]\wedge\omega= \\
& [\theta^Ad_\lambda F^\lambda_A+\theta^A_\nu(\partial^\nu_A\mathcal{L})+\cdots+\theta^A_{\nu_{s+1}\nu_s\ldots\nu_1}(\partial^{\nu_{s+1}\nu_s\ldots\nu_1}_A\mathcal{L})+\cdots]\wedge\omega= \\
& \theta^A(d_\lambda F^\lambda_A-\partial_A\mathcal{L})\wedge\omega+dL=-\delta L+dL.
\end{aligned}$$

□

Proposition 4.9 states the existence of a global finite order Lepagean equivalent $\Xi_L=\Xi+L$ of any graded Lagrangian L. Locally, one can always choose Ξ (4.21) where all functions h vanish.

5 Contact supersymmetries

A graded derivation $\upsilon\in\mathfrak{d}\mathcal{S}^0_\infty$ of the $\mathbb{R}$-ring $\mathcal{S}^0_\infty$ is said to be an infinitesimal contact supertransformation or, simply, a contact supersymmetry if the Lie derivative $\mathbf{L}_\upsilon$ preserves the ideal of contact graded forms of the BGDA $\mathcal{S}^*_\infty$ (i.e., the Lie derivative $\mathbf{L}_\upsilon$ of a graded contact form is a graded contact form).

Proposition 5.1. *With respect to the local basis $(x^\lambda, s^A_\Lambda, dx^\lambda, \theta^A_\Lambda)$ for the BGDA $\mathcal{S}^*_\infty$, any contact supersymmetry takes the form*

$$\upsilon = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + (\upsilon^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda \upsilon^A \partial^\Lambda_A), \tag{5.1}$$

where υ^λ, υ^A are local graded functions.

Proof. The key point is that, since elements of $\mathcal{C}^*_\infty$ can be identified as sections of a finite-dimensional vector bundle over X, so can elements of the $C^\infty(X)$-algebra $\mathcal{S}^*_\infty$. Moreover, any graded form is a finite composition of df, $f \in \mathcal{S}^0_\infty$. Therefore, the proof follows that of Proposition 2.3. □

The interior product $\upsilon\rfloor\phi$ and the Lie derivative $\mathbf{L}_\upsilon\phi$, $\phi \in \mathcal{S}^*_\infty$ are defined by the same formulae

$$\begin{aligned}
&\upsilon\rfloor\phi = \upsilon^\lambda\phi_\lambda + (-1)^{[\phi_A]}\upsilon^A\phi_A, \qquad \phi \in \mathcal{S}^1_\infty,\\
&\upsilon\rfloor(\phi\wedge\sigma) = (\upsilon\rfloor\phi)\wedge\sigma + (-1)^{|\phi|+[\phi][\upsilon]}\phi\wedge(\upsilon\rfloor\sigma), \qquad \phi,\sigma \in \mathcal{S}^*_\infty\\
&\mathbf{L}_\upsilon\phi = \upsilon\rfloor d\phi + d(\upsilon\rfloor\phi), \qquad \mathbf{L}_\upsilon(\phi\wedge\sigma) = \mathbf{L}_\upsilon(\phi)\wedge\sigma + (-1)^{[\upsilon][\phi]}\phi\wedge\mathbf{L}_\upsilon(\sigma).
\end{aligned}$$

as those on a graded manifold. Following the proof of Lemma 2.4, one can justify that any vertical contact supersymmetry υ (5.1) satisfies the relations

$$\upsilon\rfloor d_H\phi = -d_H(\upsilon\rfloor\phi), \tag{5.2}$$

$$\mathbf{L}_\upsilon(d_H\phi) = d_H(\mathbf{L}_\upsilon\phi), \qquad \phi \in \mathcal{S}^*_\infty. \tag{5.3}$$

Proposition 5.2. *Given a graded Lagrangian $L \in \mathcal{S}^{0,n}_\infty$, its Lie derivative $\mathbf{L}_\upsilon L$ along a contact supersymmetry υ (5.1) fulfills the first variational formula*

$$\mathbf{L}_\upsilon L = \upsilon_V\rfloor\delta L + d_H(h_0(\upsilon\rfloor\Xi_L)) + d_V(\upsilon_H\rfloor\omega)\mathcal{L}, \tag{5.4}$$

where $\Xi_L = \Xi + L$ is a Lepagean equivalent of L given by the coordinate expression (4.21).

Proof. The proof follows that of Proposition 2.5 and results from the decomposition (4.20) and the relation (5.2). □

In particular, let υ be a variational symmetry of a graded Lagrangian L, i.e., $\mathbf{L}_\upsilon L = d_H\sigma$, $\sigma \in \mathcal{S}^{0,n-1}_\infty$. Then the first variational formula (5.4) restricted to $\mathrm{Ker}\,\delta L$ leads to the weak conservation law

$$0 \approx d_H(h_0(\upsilon\rfloor\Xi_L) - \sigma). \tag{5.5}$$

Remark 5.1. Let us consider the gauge theory of principal connections on a principal bundle $P \to X$ with a structure Lie group G. These connections are represented by sections of the quotient $C = J^1P/G \to X$ [32]. This is an affine bundle coordinated by (x^λ, a^r_λ) such that, given a section A of $C \to X$, its components $A^r_\lambda = a^r_\lambda \circ A$ are coefficients of the familiar local connection form (i.e., gauge potentials). Let $J^\infty C$ be the infinite order jet manifold of $C \to X$ coordinated by $(x^\lambda, a^r_{\lambda,\Lambda})$, $0 \le |\Lambda|$, and let $\mathcal{P}^*_\infty(C)$ be the polynomial subalgebra of the GDA $\mathcal{O}^*_\infty(C)$. Infinitesimal generators of one-parameter groups of vertical automorphisms (gauge

transformations) of a principal bundle P are G-invariant vertical vector fields on $P \to X$. They are associated to sections of the vector bundle $V_GP = VP/G \to X$ of right Lie algebras of the group G. Let us consider the simple graded manifold $(X, \mathfrak{A}_{V_GY})$ constructed from this vector bundle. Its local basis is (x^λ, C^r). Let $\mathcal{C}^*_{J^rV_GY}$ be the BGDA of graded exterior forms on the graded manifold $(X, \mathfrak{A}_{J^rV_GP})$, and $\mathcal{C}^*_\infty(V_GP)$ the direct limit of the direct system (1.4) of these algebras. Then the graded product

$$\mathcal{S}^*_\infty(V_G, C) = \mathcal{C}^*_\infty(V_GP) \wedge \mathcal{P}^*_\infty(C) \tag{5.6}$$

describes gauge potentials, odd ghosts and their jets in the BRST theory. With respect to a local basis $(x^\lambda, a^r_\lambda, C^r)$ for the BGDA $\mathcal{S}^*_\infty(V_G, C)$ (5.6), the BRST symmetry is given by the contact supersymmetry

$$\begin{aligned} &\upsilon = \upsilon^r_\lambda \partial^\lambda_r + \upsilon^r \partial_r + \sum_{|\Lambda|>0} (d_\Lambda \upsilon^r_\lambda \partial^{\Lambda,\lambda}_r + d_\Lambda \upsilon^r \partial^\Lambda_r), \\ &\upsilon^r_\lambda = C^r_\lambda + c^r_{pq} a^p_\lambda C^q, \qquad \upsilon^r = -\frac{1}{2} c^r_{pq} C^p C^q, \end{aligned} \tag{5.7}$$

where c^r_{pq} are structure constants of the Lie algebra of G and ∂^λ_r, ∂_r, $\partial^{\Lambda,\lambda}_r$ ∂^Λ_r are the duals of da^r_λ, dC^r, $da^r_{\Lambda,\lambda}$ and dC^r_Λ, respectively. A remarkable peculiarity of this contact supersymmetry is that the Lie derivative $\mathbf{L}_\upsilon$ along υ (5.7) is nilpotent on the module $S^{0,*}_\infty$ of horizontal graded forms.

In a general setting, a vertical contact supersymmetry υ (5.1) is said to be nilpotent if

$$\mathbf{L}_\upsilon(\mathbf{L}_\upsilon \phi) = \sum_{|\Sigma|\geq 0, |\Lambda|\geq 0} (\upsilon^B_\Sigma \partial^\Sigma_B(\upsilon^A_\Lambda)\partial^\Lambda_A + (-1)^{[s^B][\upsilon^A]} \upsilon^B_\Sigma \upsilon^A_\Lambda \partial^\Sigma_B \partial^\Lambda_A)\phi = 0 \tag{5.8}$$

for any horizontal graded form $\phi \in S^{0,*}_\infty$.

Lemma 5.3. *A contact supersymmetry υ is nilpotent iff it is odd and the equality*

$$\mathbf{L}_\upsilon(\upsilon^A) = \sum_{|\Sigma|\geq 0} \upsilon^B_\Sigma \partial^\Sigma_B(\upsilon^A) = 0$$

holds for all υ^A.

Proof. There is the relation

$$d_\lambda \circ \upsilon^i_\Lambda \partial^\Lambda_i = \upsilon^i_\Lambda \partial^\Lambda_i \circ d_\lambda, \tag{5.9}$$

similar to (2.14). Then the lemma follows from the equality (5.8) where one puts $\phi = s^A$ and $\phi = s^A_\Lambda s^B_\Sigma$. □

Remark 5.2. A useful example of a nilpotent contact supersymmetry is the supersymmetry

$$\upsilon = \upsilon^A(x)\partial_A + \sum_{|\Lambda|>0} \partial_\Lambda \upsilon^A \partial^\Lambda_A, \tag{5.10}$$

where all υ^A are smooth real functions on X, ∂^λ_A are the duals of ds^A_Λ, but all s^A are odd.

6 Cohomology of nilpotent contact supersymmetries

Let υ be a nilpotent contact supersymmetry. Since the Lie derivative $\mathbf{L}_\upsilon$ obeys the relation (5.3), let us assume that the $\mathbb{R}$-module $\mathcal{S}^{0,*}_\infty$ of graded horizontal forms is split into a bicomplex $\{S^{k,m}\}$ with respect to the nilpotent operator $\mathbf{s}_\upsilon$ (1.5) and the total differential d_H which obey the relation

$$\mathbf{s}_\upsilon \circ d_H + d_H \circ \mathbf{s}_\upsilon = 0. \tag{6.1}$$

This bicomplex

$$d_H : S^{k,m} \to S^{k,m+1}, \qquad \mathbf{s}_\upsilon : S^{k,m} \to S^{k+1,m}$$

is graded by the form degree $0 \leq m \leq n$ and an integer $k \in \mathbb{Z}$, though it may happen that $S^{k,*} = 0$ starting from some number k. For the sake of brevity, let us call k the charge number.

For instance, the BRST bicomplex $S^{0,*}_\infty(C, V_G P)$ is graded by the charge number k which is the polynomial degree of its elements in odd variables C^r_Λ. In this case, $\mathbf{s}_\upsilon$ (1.5) is the BRST operator. Since the ghosts C^r_Λ are characterized by the ghost number 1, $k \in \mathbb{N}$, is the ghost number. The bicomplex defined by the contact supersymmetry (5.10) has the similar gradation, but taken with the sign minus (i.e., $k = 0, -1, \ldots$) because the nilpotent operator $\mathbf{s}_\upsilon$ decreases the odd polynomial degree.

Let us consider the relative and iterated cohomology of the nilpotent operator $\mathbf{s}_\upsilon$ (1.5) with respect to the total differential d_H. Recall that a horizontal graded form $\phi \in S^{*,*}$ is said to be a relative closed form, i.e., $(\mathbf{s}_\upsilon/d_H)$-closed form if $\mathbf{s}_\upsilon\phi$ is a d_H-exact form. This form is called exact if it is a sum of an $\mathbf{s}_\upsilon$-exact form and a d_H-exact form. Accordingly, we have the relative cohomology $H^{*,*}(\mathbf{s}_\upsilon/d_H)$. If a $(\mathbf{s}_\upsilon/d_H)$-closed form ϕ is also d_H-closed, it is called an iterated $(\mathbf{s}_\upsilon|d_H)$-closed form. This form ϕ is said to be exact if $\phi = \mathbf{s}_\upsilon\xi + d_H\sigma$, where ξ is a d_H-closed form. Thus, we obtain the iterated cohomology $H^{*,*}(\mathbf{s}_\upsilon|d_H)$ of the $(\mathbf{s}_\upsilon, d_H)$-bicomplex $S^{*,*}$. It is the term $E^{*,*}_2$ of the spectral sequence of this bicomplex [31]. There is an obvious isomorphism $H^{*,n}(\mathbf{s}_\upsilon/d_H) = H^{*,n}(\mathbf{s}_\upsilon|d_H)$ of relative and iterated cohomology groups on horizontal graded densities. Forthcoming Theorems 6.2 and 6.5 extend our results on iterated cohomology in [21] to an arbitrary nilpotent contact supersymmetry.

Proposition 6.1. *Let us consider the complex*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{S}^0_\infty \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty \cdots \xrightarrow{d_H} \mathcal{S}^{0,n}_\infty \xrightarrow{d_H} 0. \tag{6.2}$$

Its cohomology groups $H^{m<n}(d_H)$ equal the de Rham cohomology groups $H^{m<n}(X)$ of X, while the cohomology group $H^n(d_H)$ fulfills the relation

$$H^n(d_H)/H^n(X) = E_1. \tag{6.3}$$

Proof. The complex (6.2) differs from the short variational complex (4.1) in the last term. Therefore its cohomology $H^{m<n}(d_H)$ equals the cohomology of the complex (4.1) of the form degree $m < n$. The formula (6.3) follows from the relations: (i) $H^n(d_H) = \mathcal{S}^{0,n}_\infty/d_H(\mathcal{S}^{0,n-1}_\infty)$,

(ii) $E_1 = \mathcal{S}^{0,n}_\infty/\mathrm{Ker}\,\delta$, since δ in the complex (4.1) is an epimorphism, and (iii) $\mathrm{Ker}\,\delta/d_H(\mathcal{S}^{0,n-1}_\infty) = H^n(X)$ owing to the formula (4.19). $\Box$

Theorem 6.2. *There is an epimorphism*

$$\zeta: H^{m<n}(X) \to H^{*,m<n}(\mathbf{s}_\upsilon|d_H) \tag{6.4}$$

of the de Rham cohomology $H^m(X)$ of X of form degree less than n onto the iterated cohomology $H^{*,m<n}(\mathbf{s}_\upsilon|d_H)$.

Proof. Since a nilpotent contact supersymmetry υ is vertical, all exterior forms ϕ on X are $\mathbf{s}_\upsilon$-closed. It follows that d-cocycles on X are $(\mathbf{s}_\upsilon|d_H)$-closed. Since any d_H-exact horizontal graded form is also $(\mathbf{s}_\upsilon|d_H)$-exact, we have a morphism ζ (6.4). By virtue of Corollary 4.8 (and, equivalently, Proposition 6.1), any d_H-closed horizontal graded $(m < n)$-form ϕ is split into the sum $\phi = \varphi + d_H\xi$ (4.18) of a closed m-form φ on X and a d_H-exact graded form. Therefore, any $(\mathbf{s}_\upsilon|d_H)$-cocycle is the sum of a closed exterior form on X and a d_H-exact graded form. It follows that the morphism ζ (6.4) is an epimorphism. The kernel of the morphism ζ (6.4) consists of elements whose representatives are $\mathbf{s}_\upsilon$-exact closed exterior forms on X. $\Box$

In particular, if $X = \mathbb{R}^n$, the iterated cohomology $H^{*,0<m<n}(\mathbf{s}_\upsilon|d_H)$ is trivial in contrast with the relative ones.

For instance, since coefficients of the BRST transformation υ (5.7) consist of polynomials in ghosts C^r of non-zero degree, exterior forms on X are never $\mathbf{s}_\upsilon$-exact and, consequently, ζ (6.4) is an isomorphism. However, this is not the case of the supersymmetry (5.10).

Corollary 6.3. *If exterior forms on X are only of zero charge number, the iterated cohomology $H^{\neq 0,m<n}(\mathbf{s}_\upsilon|d_H)$ is trivial and ζ (6.4) is an epimorphism $H^{m<n}(X) \to H^{0,m<n}(\mathbf{s}_\upsilon|d_H)$.*

In particular, this is the case of the contact supersymmetries (5.7) and (5.10). Moreover, if υ is the BRST transformation (5.7), ζ in Corollary 6.3 is an isomorphism.

The bicomplex $S^{*,*}$ is a complex with respect to the total coboundary operator $\widetilde{\mathbf{s}}_\upsilon = \mathbf{s}_\upsilon + d_H$. We intend to determine the relation between the iterated cohomology $H^{*,m}(\mathbf{s}_\upsilon|d_H)$ and the total $\widetilde{\mathbf{s}}_\upsilon$-cohomology $H^*(\widetilde{\mathbf{s}}_\upsilon)$ of the bicomplex $S^{*,*}$.

There exists the morphism

$$\gamma: H^{<n}(X) \to H^*(\widetilde{\mathbf{s}}_\upsilon) \tag{6.5}$$

of the de Rham cohomology $H^{<n}(X)$ of X of form degree $< n$ to the total cohomology $H^*(\widetilde{\mathbf{s}}_\upsilon)$ similar to the morphism (6.4). The morphism γ (6.5) associates to a closed m-form ϕ of charge number k the $\widetilde{s}_\upsilon$-cocycle of total charge $(k+m)$ whose representative is ϕ. Its kernel consists of elements whose representatives are $\widetilde{\mathbf{s}}_\upsilon$-exact closed exterior forms on X.

Theorem 6.4. *There is a monomorphism of the iterated cohomology $H^{*,m<n}(\mathbf{s}_\upsilon|d_H)$ to the total cohomology $H^*(\widetilde{\mathbf{s}}_\upsilon)$ which associates to an iterated (k,m)-cocycle $[\phi]$ the $\widetilde{s}_\upsilon$-cocycle of total charge $(k+m)$ represented by the same graded form ϕ.*

Proof. Any $(\mathbf{s}_\upsilon|d_H)$-cocycle, by definition, is a $\widetilde{s}_\upsilon$-cocycle. Using the formula (4.18) and the fact that exterior forms on X are s_υ-closed, one can show that: (i) any $(\mathbf{s}_\upsilon|d_H)$-coboundary is also a d_H-coboundary and, consequently, a $\widetilde{s}_\upsilon$-coboundary, and (ii) if a $(\mathbf{s}_\upsilon|d_H)$-cocycle is a $\widetilde{s}_\upsilon$-coboundary, then it is a $(\mathbf{s}_\upsilon|d_H)$-coboundary. This proves the statement. □

Turn now to the iterated cohomology $H^{*,n}(\mathbf{s}_\upsilon|d_H)$. This requires careful analysis since Proposition 6.1 implies that the cohomology $H^n(d_H)$ of the complex (6.2) fails to equal the de Rham cohomology $H^n(X)$ of X.

Theorem 6.5. *Put $\overline{H}^* = H^*(\widetilde{\mathbf{s}}_\upsilon)/\mathrm{Im}\,\gamma$, where the asterisk means the total charge. There is an isomorphism*

$$H^{*,n}(\mathbf{s}_\upsilon|d_H)/\overline{H}^* = \mathrm{Ker}\,\gamma. \tag{6.6}$$

Proof. The proof falls into the following three steps.

(i) First, we show the existence of a morphism

$$\eta : H^{*,n}(\mathbf{s}_\upsilon|d_H) \to \mathrm{Ker}\,\gamma \tag{6.7}$$

from the iterated cohomology group $H^{*,n}(\mathbf{s}_\upsilon|d_H)$ to $\mathrm{Ker}\,\gamma$. Consider a horizontal graded n-form ϕ_n which is $(\mathbf{s}_\upsilon|d_H)$-closed. Then, by definition, $\mathbf{s}_\upsilon\phi_n$ is d_H-exact, i.e.,

$$\mathbf{s}_\upsilon\phi_n + d_H\phi_{n-1} = 0. \tag{6.8}$$

Acting on this equality by $\mathbf{s}_\upsilon$, we observe that $\mathbf{s}_\upsilon\phi_{n-1}$ is a d_H-closed graded form, i.e.,

$$\mathbf{s}_\upsilon\phi_{n-1} + d_H\phi_{n-2} = \varphi_{n-1}, \tag{6.9}$$

where φ_{n-1} is a closed $(n-1)$-form on X in accordance with Corollary 4.8. Since $\mathbf{s}_\upsilon\varphi_{n-1} = 0$, an action of $\mathbf{s}_\upsilon$ on the equation (6.9) shows that $\mathbf{s}_\upsilon\phi_{n-2}$ is a d_H-closed graded form, i.e.,

$$\mathbf{s}_\upsilon\phi_{n-2} + d_H\phi_{n-3} = \varphi_{n-2},$$

where φ_{n-2} is a closed $(n-2)$-form on X. Iterating the arguments, one comes to the system of equations

$$\mathbf{s}_\upsilon\phi_{n-k} + d_H\phi_{n-k-1} = \varphi_{n-k}, \qquad 0 \le k < n, \qquad \mathbf{s}_\upsilon\phi_0 = \varphi_0 = \mathrm{const}, \tag{6.10}$$

which can be assembled into descent equations

$$\widetilde{\mathbf{s}}_\upsilon\widetilde{\phi} = \widetilde{\varphi}, \tag{6.11}$$

$$\widetilde{\phi} = \phi_n + \phi_{n-1} + \cdots + \phi_0, \qquad \widetilde{\varphi} = \varphi_{n-1} + \cdots + \varphi_0. \tag{6.12}$$

Thus, any $(\mathbf{s}_\upsilon|d_H)$-closed horizontal graded form defines descent equations (6.11) whose right-hand sides $\widetilde{\varphi}$ are closed exterior forms on X such that their de Rham classes belong to the kernel $\mathrm{Ker}\,\gamma$ of the morphism (6.5). For the sake of brevity, let us denote these descent equations by $\langle\widetilde{\varphi}\rangle$. Accordingly, we say that a horizontal graded form $\widetilde{\phi}$ (6.12) is a solution of descent equations $\langle\widetilde{\varphi}\rangle$ (6.12). Descent equations defined by a $(\mathbf{s}_\upsilon|d_H)$-closed horizontal

graded form ϕ_n are not unique. Let $\widetilde{\phi}'$ be another solution of another set of descent equations $\langle\widetilde{\varphi}'\rangle$ such that $\phi_n = \phi'_n$. Let us denote $\Delta\phi_k = \phi_k - \phi'_k$ and $\Delta\varphi_k = \varphi_k - \varphi'_k$. Then the equations (6.8) lead to the equation $d_H(\Delta\phi_{n-1}) = 0$. It follows from Corollary 4.8 that

$$\Delta\phi_{n-1} = d_H\xi_{n-2} + \alpha_{n-1}, \tag{6.13}$$

where α_{n-1} is a closed $(n-1)$-form on X. Accordingly, the equation (6.10) leads to the equation

$$\mathbf{s}_\upsilon(\Delta\phi_{n-1}) + d_H(\Delta\phi_{n-2}) = \Delta\varphi_{n-1}.$$

Substituting the equality (6.13) into this equation and bearing in mind the relation (6.1), we obtain the equality

$$d_H(-\mathbf{s}_\upsilon\xi_{n-2} + \Delta\phi_{n-2}) = \Delta\varphi_{n-1}.$$

It follows that

$$\Delta\phi_{n-2} = \mathbf{s}_\upsilon\xi_{n-2} + d_H\xi_{n-3} + \alpha_{n-2}, \qquad \Delta\varphi_{n-1} = d\alpha_{n-2}$$

where α_{n-2} is an exterior form on X. Iterating the arguments, one comes to the relations

$$\Delta\phi_{n-k} = \mathbf{s}_\upsilon\xi_{n-k} + d_H\xi_{n-k-1} + \alpha_{n-k}, \qquad \Delta\varphi_{n-k} = d\alpha_{n-k-1}, \qquad 1 < k < n, \tag{6.14}$$

where α_{n-k-1} are exterior forms on X and, finally, to the equalities $\Delta\phi_0 = 0$, $\Delta\varphi_0 = 0$. Then it is easily justified that

$$\widetilde{\phi} - \widetilde{\phi}' = \widetilde{\mathbf{s}}_\upsilon\widetilde{\sigma} + \widetilde{\alpha}, \qquad \widetilde{\sigma} = \xi_{n-2} + \cdots\xi_1, \tag{6.15}$$

$$\widetilde{\varphi} - \widetilde{\varphi}' = d\widetilde{\alpha}, \qquad \widetilde{\alpha} = \alpha_{n-1} + \cdots + \alpha_1. \tag{6.16}$$

It follows that right-hand sides of any two descent equations defined by a $(\mathbf{s}_\upsilon|d_H)$-closed horizontal graded form ϕ_n differ from each other in an exact form on X. Moreover, let ϕ_n and ϕ'_n be representatives of the same iterated cohomology class in $H^{*,n}(\mathbf{s}_\upsilon|d_H)$, i.e., $\phi'_n = \phi_n + \mathbf{s}_\upsilon\psi + d_H\beta$, where ψ is d_H-closed. Let ϕ_n provide a solution $\widetilde{\phi}$ of a descent equation $\langle\widetilde{\varphi}\rangle$. Then ϕ'_n defines a solution $\widetilde{\phi}' = \widetilde{\phi} + \widetilde{\mathbf{s}}_\upsilon(\psi + \beta)$ of the same descent equation. Thus, the assignment $\phi_n \mapsto \langle\widetilde{\varphi}\rangle$ yields the desired morphism η (6.7).

(ii) Let us show that the morphism η (6.7) is an epimorphism. Let $\widetilde{\varphi}$ be a closed exterior form on X whose de Rham cohomology class belongs to $\operatorname{Ker}\gamma$. Let $\widetilde{\varphi} = \varphi_{n-1} + \cdots + \varphi_0$ be its decomposition in k-forms φ_k, $k = 1, \ldots, n-1$. Then the family of exterior forms (φ_k) yields a system of the equations (6.10) which can be assembled into the descent equations $\langle\widetilde{\varphi}\rangle$ (6.11). Its solution $\widetilde{\phi}$ exists because $\widetilde{\varphi} \in \operatorname{Ker}\gamma$. Let $\widetilde{\varphi}'$ differ from $\widetilde{\varphi}$ in an exact form, i.e., let the relation (6.16) hold. Then any solution $\widetilde{\phi}$ of the equation $\langle\widetilde{\varphi}\rangle$ yields a solution $\widetilde{\phi}' = \widetilde{\phi} - \widetilde{\alpha}$ (6.15) of the equation $\langle\widetilde{\varphi}'\rangle$ such that $\phi'_n = \phi_n$. It follows that the morphism η (6.7) is an epimorphism.

(iii) The kernel of the morphism η (6.7) is represented by $(\mathbf{s}_\upsilon|d_H)$-closed horizontal graded forms ϕ_n which yield homogeneous descent equations

$$\widetilde{\mathbf{s}}_\upsilon\widetilde{\phi} = 0. \tag{6.17}$$

Let us define an epimorphism of the total cohomology $H^*(\widetilde{\mathbf{s}}_v)$ onto $\operatorname{Ker}\eta$. For this purpose, let us associate to each $\widetilde{s}_v$-cocycle $\widetilde{\phi}$ its higher term ϕ_n. The latter defines homogeneous descent equations (6.17) whose solution is $\widetilde{\phi}$, i.e., $\phi_n \in \operatorname{Ker}\eta$. Let $\widetilde{\phi} = \widetilde{s}_v\widetilde{\psi}$ be a $\widetilde{s}_v$-coboundary. Its higher term ϕ_n takes the form $\phi_n = s_v\psi_n + d_H\psi_{n-1}$, i.e., it is an iterated coboundary. It follows that the assignment $\widetilde{\phi} \mapsto \phi_n$ provides the desired epimorphism $\tau : H^*(\widetilde{\mathbf{s}}_v) \to \operatorname{Ker}\eta$. The kernel of this epimorphism is represented by solutions $\widetilde{\phi}$ of the descent equation (6.17) whose higher term vanishes. Following item (i), one can easily show that these solutions take the form $\widetilde{\phi} = \widetilde{\mathbf{s}}_v\sigma + \widetilde{\alpha}$, where $\widetilde{\alpha}$ is a closed exterior form on X of form degree $< n$. Cohomology classes of these solutions exhaust the image of the morphism γ (6.5), i.e., $\operatorname{Im}\gamma = \operatorname{Ker}\tau$. □

In particular, if the morphism γ (6.5) is a monomorphism (i.e., no non-exact closed exterior form on X is $\widetilde{\mathbf{s}}_v$-exact), the isomorphism (6.6) gives the isomorphism

$$H^{*,n}(\mathbf{s}_v|d_H) = H^*(\widetilde{\mathbf{s}}_v)/H^{<n}(X).$$

For instance, this is the case of the BRST transformation (5.7) [21].

7 Conclusion

In the present work, we follow the algebraic topological approach to describing Lagrangian field theories in terms of the variational bicomplex. This enables us to extend the cohomology analysis of Lagrangian BRST theory on $\mathbb{R}^n$ to a generic contact supersymmetry on an arbitrary manifold X. Since only vector and affine bundles over X are involved, the corresponding cohomology characteristic is represented by the de Rham cohomology of X. In a general case of nilpotent contact supersymmetry $\mathbf{s}_v$, its contribution however is not trivial because exterior forms on X are $\mathbf{s}_v$-closed, but need not be $\mathbf{s}_v$-exact. For instance, this contribution is given by the kernel of the morphism γ (6.5) in Theorem 6.5. Our analysis seems important for BV quantization of field systems with non-contractible topologies, e.g., gravitation theory and topological field models. We also bear in mind the extension of BV quantization to field systems where parameters of gauge transformations may depend on field variables and their derivatives [18]. For instance, this is the case of spinor fields in gauge gravitation theory [37].

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8 Appendixes

Appendix A. Since Y is a strong deformation retract of any finite order jet manifold J^rY, the de Rham cohomology of the GDA $\mathcal{O}^*_\infty$ is easily proved to equal the de Rham cohomology $H^*(Y)$ of Y in accordance with Theorem 8.3 [2]. However, we must enlarge $\mathcal{O}^*_\infty$ in order to find its d_H- and δ-cohomology.
Outline of proof of Theorem 2.1. One starts from the algebraic Poincaré lemma [35, 43].

Lemma 8.1. *If Y is a contractible bundle $\mathbb{R}^{n+p} \to \mathbb{R}^n$, the variational bicomplex (2.4) is exact.*

For instance, the homotopy operators for d_V, d_H, δ and ϱ are given by the formulae (5.72), (5.109), (5.84) in [35] and (4.5) in [43], respectively.

Let $\mathfrak{O}^*_r$ be the sheaf of germs of exterior forms on the r-order jet manifold J^rY, and let $\overline{\mathfrak{O}}^*_r$ be its canonical presheaf. There is the direct system of presheaves

$$\overline{\mathfrak{O}}^*_X \xrightarrow{\pi^*} \overline{\mathfrak{O}}^*_0 \xrightarrow{\pi^{1*}_0} \overline{\mathfrak{O}}^*_1 \cdots \xrightarrow{\pi^{r}_{r-1}{}^*} \overline{\mathfrak{O}}^*_r \longrightarrow \cdots .$$

Its direct limit $\overline{\mathfrak{O}}^*_\infty$ is a presheaf of GDAs on the infinite order jet manifold $J^\infty Y$. Let $\mathfrak{T}^*_\infty$ be the sheaf of GDAs on $J^\infty Y$ constructed from the presheaf $\overline{\mathfrak{O}}^*_\infty$, i.e., $\mathfrak{T}^*_\infty$ is the shef of germs of $\overline{\mathfrak{O}}^*_\infty$ (we follow the terminology of [28]). The structure module $\Gamma(\mathfrak{T}^*_\infty)$ of sections of $\mathfrak{T}^*_\infty$ is a GDA such that, given an element $\phi \in \Gamma(\mathfrak{T}^*_\infty)$ and a point $z \in J^\infty Y$, there exist an open neighbourhood U of z and an exterior form $\phi^{(k)}$ on some finite order jet manifold J^kY so that $\phi|_U = \pi^{\infty *}_k \phi^{(k)}|_U$. In particular, there is the monomorphism $\mathcal{O}^*_\infty \to \Gamma(\mathfrak{T}^*_\infty)$. The fact that the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $\Gamma(\mathfrak{T}^0_\infty)$ [42], enables one to obtain d_H- and δ-cohomology of $\Gamma(\mathfrak{T}^*_\infty)$ as follows [1, 2, 41, 42].

The sheaf $\mathfrak{T}^*_\infty$ is split into the bicomplex $\mathfrak{T}^{*,*}_\infty$. Let us consider its variational subcomplex and the complexes of sheaves of contact forms

$$0 \to \mathbb{R} \to \mathfrak{T}^0_\infty \xrightarrow{d_H} \mathfrak{T}^{0,1}_\infty \cdots \xrightarrow{d_H} \mathfrak{T}^{0,n}_\infty \xrightarrow{\delta} \mathfrak{E}_1 \xrightarrow{\delta} \mathfrak{E}_2 \longrightarrow \cdots, \qquad \mathfrak{E}_k = \varrho(\mathfrak{T}^{k,n}_\infty), \tag{8.1}$$

$$0 \to \mathfrak{T}^{k,0}_\infty \xrightarrow{d_H} \mathfrak{T}^{k,1}_\infty \cdots \xrightarrow{d_H} \mathfrak{T}^{k,n}_\infty \xrightarrow{\varrho} \mathfrak{E}_k \to 0, \tag{8.2}$$

together with complexes of their structure modules

$$0 \to \mathbb{R} \to \Gamma(\mathfrak{T}^0_\infty) \xrightarrow{d_H} \Gamma(\mathfrak{T}^{0,1}_\infty) \cdots \xrightarrow{d_H} \Gamma(\mathfrak{T}^{0,n}_\infty) \xrightarrow{\delta} \Gamma(\mathfrak{E}_1) \xrightarrow{\delta} \Gamma(\mathfrak{E}_2) \longrightarrow \cdots, \tag{8.3}$$

$$0 \to \Gamma(\mathfrak{T}^{k,0}_\infty) \xrightarrow{d_H} \Gamma(\mathfrak{T}^{k,1}_\infty) \cdots \xrightarrow{d_H} \Gamma(\mathfrak{T}^{k,n}_\infty) \xrightarrow{\varrho} \Gamma(\mathfrak{E}_k) \to 0. \tag{8.4}$$

By virtue of Lemma 8.1 and Theorem 8.3, the complexes (8.1) – (8.2) are exact. Since $\mathfrak{T}^{*,*}_\infty$ are sheaves of $\Gamma(\mathfrak{T}^0_\infty)$-modules, they are fine. The sheaves $\mathfrak{E}_k$ are also proved to be fine [21, 41]. Consequently, all sheaves, except $\mathbb{R}$, in the complexes (8.1) – (8.2) are acyclic. Therefore, these complexes are the resolutions of the constant sheaf $\mathbb{R}$ and the zero sheaf over $J^\infty Y$, respectively. In accordance with the abstract de Rham theorem ([28], Theorem 2.12.1), cohomology of the complex (8.3) equals the cohomology of $J^\infty Y$ with coefficients in $\mathbb{R}$, while the complex (8.4) is exact. Since Y is a strong deformation retract of $J^\infty Y$ [2, 22], cohomology of the complex (8.3) is isomorphic to the de Rham cohomology of Y.

Note that, in order to prove the exactness of the complex (8.4), one can use a minor generalization of the above mentioned abstract de Rham theorem (see Appendix D), and need not justify the acyclicity of the sheaves $\mathfrak{E}_k$ [42].

Finally, the subalgebra $\mathcal{O}^*_\infty \subset \Gamma(\mathfrak{T}^*_\infty)$ is proved to have the same d_H- and δ-cohomology as $\Gamma(\mathfrak{T}^*_\infty)$ [21, 41]. Similarly, one can show that, restricted to $\mathcal{O}^{k,n}_\infty$, the operator ϱ remains exact. □

The following is a corollary of item (i) of Theorem 2.1 (cf. Corollary 4.8).

Corollary 8.2. *Every d_H-closed form $\phi \in \mathcal{O}^{0,m<n}$ is the sum*

$$\phi = h_0\psi + d_H\xi, \qquad \xi \in \mathcal{O}^{0,m-1}_\infty, \tag{8.5}$$

where ψ is a closed m-form on Y. Every δ-closed Lagrangian $L \in \mathcal{O}^{0,n}_\infty$ is the sum

$$L = h_0\psi + d_H\xi, \qquad \xi \in \mathcal{O}^{0,n-1}_\infty, \tag{8.6}$$

where ψ is a closed n-form on Y.

Note that the formulae (8.5) – (8.6) were obtained in [1] by computing cohomology of the fixed order variational sequence, but the proof of the local exactness of this sequence requires rather sophisticated *ad hoc* techniques.

Appendix B. Let us consider the open surjection $\pi^\infty : J^\infty Y \to X$ and the direct image $\pi^*_\infty \mathfrak{T}^*_\infty$ on X of the sheaf $\mathfrak{T}^*_\infty$ of exterior forms on $J^\infty Y$. Its stalk at a point $x \in X$ consists of the equivalence classes of sections of the sheaf $\mathfrak{T}^*_\infty$ which coincide on the inverse images $(\pi^\infty)^{-1}(U_x)$ of open neighbourhoods U_x of X. Since $(\pi^\infty)^{-1}(U_x)$ is the infinite order jet manifold of sections of the fiber bundle $\pi^{-1}(U_x) \to X$, every point $x \in X$ has a base of open neighbourhoods $\{U_x\}$ such that the sheaves $\mathfrak{T}^{*,*}_\infty$ and $\mathfrak{E}_k$ in the proof of Theorem 2.1 are acyclic on the inverse images $(\pi^\infty)^{-1}(U_x)$ of these neighbourhoods. Then, in accordance with the Leray theorem [23], cohomology of $J^\infty Y$ with coefficients in the sheaves $\mathfrak{T}^{*,*}_\infty$ and $\mathfrak{E}_k$ is isomorphic to that of X with coefficients in their direct images $\pi^*_\infty \mathfrak{T}^{*,*}_\infty$ and $\pi^*_\infty \mathfrak{E}_k$, i.e., the sheaves $\pi^*_\infty \mathfrak{T}^{*,*}_\infty$ and $\pi^*_\infty \mathfrak{E}_k$ over X are acyclic. Let $Y \to X$ be an affine bundle. Then X is a strong deformation retract of $J^\infty Y$. In this case, the inverse images $(\pi^\infty)^{-1}(U_x)$ of contractible neighbourhoods U_x are contractible and $\pi^\infty_* \mathbb{R} = \mathbb{R}$. Then, by virtue of Lemma 8.1, the variational bicomplex $\mathfrak{T}^*_\infty$ of sheaves over $(\pi^\infty)^{-1}(U_x)$ is exact, and the variational bicomplex $\pi^\infty_* \mathfrak{T}^*_\infty$ of sheaves over X is so. There is an $\mathbb{R}$-algebra isomorphism of the GDA of sections of the sheaf $\pi^\infty_* \mathfrak{T}^*_\infty$ over X to the GDA $\Gamma(\mathfrak{T}^*_\infty)$. Thus, the GDA $\Gamma(\mathfrak{T}^*_\infty)$ and its subalgebra $\mathcal{O}^*_\infty$ can be regarded as algebras of sections of a sheaf over X.

Appendix C. Let us associate to each open subset $U \subset X$ the bigraded algebra $\mathcal{S}^*_U$ of elements of the $C^\infty(X)$-algebra $\mathcal{S}^*_\infty$ whose coefficients are restricted to U. These algebras make up a presheaf

$$\{\mathcal{S}^*_U, r^U_V \mid r^U_V : \mathcal{S}^*_U \to \mathcal{S}^*_V\} \tag{8.7}$$

over X. Let $\mathfrak{S}^*_\infty$ be the sheaf constructed from this presheaf. Its stalk $\mathfrak{S}^*_x$ at a point $x \in X$ is the direct limit of the direct system of $\mathbb{R}$-modules $\{\mathcal{S}^*_U, r^U_V\}$, indexed by the directed set of open neighbourhoods U of x. This stalk consists of the germs of elements of $\mathcal{S}^*_\infty$ at x, i.e., elements of the presheaf (8.7) are identified if their restrictions (namely, the restrictions of their coefficients) to some open neighbourhood of x coincide with each other. Let $\mathcal{S}^*_c$ be the subalgebra of the BGDA $\mathcal{S}^*_\infty$ on $X = \mathbb{R}^n$ which consists of elements with constant coefficients. Then $\mathfrak{S}^*_x$ is the stalk of germs of $\mathcal{S}^*_c$-valued functions on X. It is a bigraded algebra isomorphic to the tensor product $C^\infty_x \otimes_\mathbb{R} \mathcal{S}^*_c$ of the $\mathbb{R}$-algebra C^∞_x of the germs of smooth real functions on X at x and the $\mathbb{R}$-algebra $\mathcal{S}^*_c$. This stalk is naturally decomposed into C^∞_x-modules $\mathfrak{S}^{k,m}_x$ of the germs of graded (k,m)-forms on X.

Let the common symbol Δ stand for all the operators (d,d_H, d_V, ϱ and δ) on the BGDA $\mathcal{S}^*_\infty$. It is an $\mathbb{R}$-module morphism whose restrictions Δ_U to $\mathcal{S}^*_U$ intertwined by the restriction morphisms r^U_V (8.7) constitute the direct system of morphisms

$$\{\Delta_U, r^U_V \mid r^U_V \circ \Delta_U = \Delta_V \circ r^U_V\}, \tag{8.8}$$

indexed by the directed set of open neighbourhoods U of x. Its direct limit Δ_x is an $\mathbb{R}$-module morphism of the stalk $\mathfrak{S}^*_x$. The properties of the direct limit of morphisms are summarized by the following theorem [33].

Theorem 8.3. *The direct limit of a direct system of complexes* $\{C^*_i, i \in I\}$ *is a complex whose cohomology is the direct limit of that of the complexes* C^*_i.

It follows that the stalk $\mathfrak{S}^*_x$ is a BGDA which contains the complexes corresponding to the subcomplexes of the variational bicomplex (3.6) of the BGDA $\mathcal{S}^*_\infty$. Since, $d_x = d_{Hx} + d_{Vx}$ and the operators d_x, d_{Hx}, d_{Vx} are nilpotent, we have a bicomplex $\mathfrak{S}^{*,*}_x$. Moreover, the vertical differential d_{Vx} on $\mathfrak{S}^*_x = C^\infty_x \otimes_\mathbb{R} \mathcal{S}^*_c$ comes from the operator d_V on $\mathcal{S}^*_c$. Therefore, $\delta_x = \varrho_x \circ d_{Vx}$ and the subcomplexes of $\mathfrak{S}^*_x$ can be assembled into the variational bicomplex. Accordingly, the sheaf $\mathfrak{S}^*_\infty$ of germs of graded exterior forms on X constitutes the variational bicomplex

$$(\mathcal{O}^*_X, \mathfrak{S}^{*,*}_\infty, \mathfrak{E}_k = \varrho(\mathfrak{S}^{k,n}_\infty); d, d_H, d_V, \varrho, \delta), \tag{8.9}$$

where the operators on $\mathfrak{S}^*_\infty$ are denoted by the same symbols as those on the BGDA $\mathcal{S}^*_\infty$.

Let $\Gamma(\mathfrak{S}^*_\infty)$ be the bigraded algebra of sections of the sheaf $\mathfrak{S}^*_\infty$. Given an arbitrary section s of $\Gamma(\mathfrak{S}^*_\infty)$, there exists an open neighbourhood U of each point $x \in X$ such that $s|_U$ is an element of the presheaf (8.7). It follows that $\Gamma(\mathfrak{S}^*_\infty)$ can be provided with the same operators d, d_H, d_V, ϱ and δ as the BGDA $\mathcal{S}^*_\infty$ which make it into the $\mathbb{Z}_2$-graded variational bicomplex, analogous to that of $\mathcal{S}^*_\infty$. The homomorphism $\mathcal{S}^*_\infty \to \Gamma(\mathfrak{S}^*_\infty)$ is a monomorphism.

Appendix D. We quote the following minor generalization of the abstract de Rham theorem ([28], Theorem 2.12.1) [20, 22, 42]. Let

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \qquad p > 1,$$

be an exact sequence of sheaves of abelian groups over a paracompact topological space Z, where the sheaves S_q, $0 \le q < p$, are acyclic, and let

$$0 \to \Gamma(Z,S) \xrightarrow{h_*} \Gamma(Z,S_0) \xrightarrow{h^0_*} \Gamma(Z,S_1) \xrightarrow{h^1_*} \cdots \xrightarrow{h^{p-1}_*} \Gamma(Z,S_p) \xrightarrow{h^p_*} \Gamma(Z,S_{p+1}) \tag{8.10}$$

be the corresponding cochain complex of structure groups of these sheaves.

Theorem 8.4. *The* q*-cohomology groups of the cochain complex (8.10) for* $0 \le q \le p$ *are isomorphic to the cohomology groups* $H^q(Z,S)$ *of* Z *with coefficients in the sheaf* S.

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